ON SOME EXTREMAL FUNCTIONS AND THEIR APPLICATIONS IN THE THEORY OF ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES[1]

BY

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1. **Introduction.** Let E be a bounded closed set in the space C^n of n-complex variables $z = (z_1, ..., z_n)$. Let $b(z)$ be a real function defined and bounded on E. In the following we define an extremal function $\Phi(z, E, b)$, $z \in C^n$, depending on E and b. For this purpose we introduce a triangular array of extremal points $\{\gamma_k^{(v)}\}$ of E. In the case that $b(z)$ is lower semicontinuous, the formal definition of the points $\gamma_k^{(v)}$ is analogous to the definition of Fekete-Leja's point of a plane set. In the case that E is in C^1 and $b(z) \equiv 0$, the points $\gamma_k^{(v)}$ are exactly Fekete's points of E (see (5.3')).

In the case of one complex variable, the function $\log \Phi(z, E, 0)$ is a generalized Green's function for the unbounded component of CE with pole at ∞. It is well known that the Green's function plays the primary role in the theory of interpolation and approximation of holomorphic functions of one variable by polynomials (see [27]). It turns out that the function $\Phi(z, E, 0)$, $z \in C^n$, also plays a quite similar role in the theory of interpolation and approximation of holomorphic functions of several variables by polynomials. For instance, one can obtain the Bernstein-Walsh inequality

$$|P_v(z)| \leq M\Phi^v(z, E, 0), \quad z \in C^n, \quad M = \max_{z \in E} |P_v(z)|$$

$P_v(z)$ being an arbitrary polynomial of order v, $v = 0, 1, \ldots$. This inequality is useful in the proof of the following theorem: *If $\Phi(z, E, 0)$ is continuous in C^n and E_R is given by*

$$E_R = \{z \mid \Phi(z, E, 0) < R\}, \quad R > 1,$$

then the necessary and sufficient condition that function $f(z)$ be holomorphic in E_R and not continuable to holomorphic (single-valued) function in any $E_{R'}$, $R' > R$, is that

$$(*) \qquad \limsup_{v \to \infty} \sqrt[v]{\max_{z \in E} |f(z) - \pi_v(z)|} = \frac{1}{R},$$

where $\pi_v(z)$ denotes a polynomial of order v of the Tchebycheff best approximation to $f(z)$ on E.

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We also show that polynomials $\mathscr{L}_\nu(z,f)$ found by interpolation of $f(z)$ at the extremal points of E (with respect to $b(z) \equiv 0$) converge maximally to $f(z)$.

We prove that if E is a Cartesian product of plane sets $E_1, E_2, \ldots, E_n$, any one of which has positive logarithmic capacity, then

$$\Phi(z,E,0) \equiv \max \{\Phi(z_1,E_1,0), \ldots, \Phi(z_n,E_n,0)\}, \qquad z \in C^n.$$

This equality implies that approximation or interpolation by polynomials to the function $f(z) = f(z_1, \ldots, z_n)$ holomorphic in the Cartesian product of plane sets reduces, in principle, to approximation or interpolation in each variable separately.

For instance, if $E = E_1 \times \ldots \times E_n$, then (*) is a necessary and sufficient condition that the function $f(z)$ be holomorphic in the Cartesian product $E_{1R} \times E_{2R} \times \ldots \times E_{nR}$, where $E_{kR} = \{z_k \,|\, \Phi(z_k,E_k,0) < R\}$. In the case that E_k, $k = 1, 2, \ldots, n$, is a line segment, this fact has been proved by a different method in [17].

In the last section we prove that if $\Phi(z,E,0)$ is finite at any point of C^n, then there exists a limit

(**) $$u(z,E,b) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \log \frac{\Phi(z,E,\lambda b)}{\Phi(z,E,0)}, \qquad z \in C^n.$$

If E is a plane set of positive capacity, then the function $u(z,E,b)$ is harmonic in CE. In the case that E is a Jordan curve and $b(z)$ is continuous, the function $u(z,E,b)$ has been proved in [8] and [13] to be a solution of the Dirichlet problem for the interior of E with boundary values $b(z)$. This result has been generalized in [21] as follows. If E is the boundary of a domain D which contains the point ∞ in its interior, the function $u(z,E,b)$ is a generalized solution of the Dirichlet problem for any component of CE with boundary values $b(z)$ (continuous or not)

In the last section of this paper the connection of $u(z,E,b)$ with Bremermann's solution (see [4]) of the Dirichlet problem for plurisubharmonic functions in C^n and with domains of uniform convergency of Hartogs' series has been established. In particular, we have proved the following. If E is a Šilov boundary with respect to polynomials of a bounded domain D such that there exists a decreasing sequence of domains of holomorphy $\{D_\nu\}$ convergent to $\bar{D}$,

$$D_\nu \supset D_{\nu+1} \supset \bar{D}, \qquad \nu = 1, 2, \ldots, \qquad D_\nu \to \bar{D},$$

and if any function holomorphic in $\bar{D}$ can be approximated by polynomials uniformly in $\bar{D}$, then the function

$$u^*(z,E,b) = \limsup_{z' \to z} u(z',E,b), \qquad z \in D,$$

is an upper envelope of all functions $V(z)$ plurisubharmonic in $\bar{D}$ which are less or equal to $b(z)$ on E.

Roughly speaking, this means that for bounded polynomially convex domains the function $u^*(z,E,b)$ is equal to Bremermann's function.

The functions $\log \Phi^*(z, E, b)$, for $z \in C^n$, and $u^*(z, E, b)$, for $z \in D$, are plurisubharmonic and therefore are members of some of Bergman's extended classes [2; 3]. In the case of C^1, the functions $\log \Phi(z, E, b)$ and $u(z, E, b)$ are harmonic at any finite point outside of E and therefore are continuous there. The natural question of the continuity of $\Phi^*(z, E, b)$ or $u^*(z, E, b)$, $z \in C^n$ (or of the generalized solution of Dirichlet's problem for plurisubharmonic functions) is still open. One knows that in the case of C^1 there is also a very simple relation between Green's function and Bergman's kernel function. We do not know whether any relation between $\log \Phi^*(z, E, 0)$ and the kernel function of several complex variables can be established.

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2. Lagrange interpolation formulae for polynomials of n complex variables([2]).

Let $P_\nu(z) = P_\nu(z_1, \ldots, z_n)$ be a polynomial of degree ν

$$(1) \qquad P_\nu(z) \quad = \sum_{k_1 + \ldots + k_n \le \nu} a_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n}.$$

We shall always assume that a polynomial of degree ν is also of degree ν', $\nu' \ge \nu$. Let

$$k_{1l}, \ldots, k_{nl}, \qquad l = 1, 2, \ldots, C_{\nu+n, n}$$

denote the sequence of all solutions in nonnegative integers of the inequality $k_1 + \ldots + k_n \le \nu$. Then $P_\nu(z)$ may be written in the form

$$(2) \qquad P_\nu(z) = \sum_{l=1}^{\nu_*} a_{k_{1l} \cdots k_{nl}} z_1^{k_{1l}} \ldots z_n^{k_{nl}}, \quad \nu_* =^{df} C_{\nu+n, n}.$$

Let $p^{(\nu)} = \{p_1, p_2, \ldots, p_{\nu_*}\}$ be a system of ν_* points

$$(3) \qquad p_i = (z_{1i}, \ldots, z_{ni}), \qquad i = 1, 2, \ldots, \nu_*$$

such that the determinant

$$(4) \qquad V(p^{(\nu)}) = \det[z_{1i}^{k_{1l}} z_{2i}^{k_{2l}} \ldots z_{ni}^{k_{nl}}], \quad (i, l = 1, 2, \ldots, \nu_*),$$

is different from zero. We observe that $V(p^{(\nu)})$ is a determinant of the system of linear equations

$$(5) \qquad \sum_{l=1}^{\nu_*} a_{k_{1l} k_{2l} \ldots k_{nl}} z_{1i}^{k_{1l}} z_2^{k_{2l}} \ldots z^{k_{nl}} = b_i, \qquad i = 1, 2, \ldots, \nu_*,$$

where $a_{k_{1l} k_{2l} \ldots k_{nl}}$, $l = 1, 2, \ldots, \nu_*$, are unknowns. Therefore, there is exactly one polynomial $P(z)$ of degree ν which takes the value b_i at the point p_i of system $p^{(\nu)}$.

A system $p^{(\nu)} = \{p_1, \ldots, p_{\nu_*}\}$ for which $V(p^{(\nu)}) \ne 0$ will be called *unisolvent* with respect to polynomials of degree ν, or simply unisolvent system of order ν.

Let us replace the ith row of the determinant (4) by the row

$$[z_1^{k_{11}} z_2^{k_{21}} \ldots z_n^{k_{n1}}, \; z_1^{k_{12}} z_2^{k_{22}} \ldots z_n^{k_{n2}} \quad, \ldots, \; z_1^{k_{1\nu_*}} z_2^{k_{2\nu_*}} \ldots z_n^{k_{n\nu_*}}].$$

([2]) Concerning various interpolation formulas in C^n see for instance [22; 24].

We shall obtain a new determinant (4), say $V_i(z,p^{(v)})$, which corresponds to the system of points

$$\{p_1, p_2, \ldots, p_{i-1}, z, p_{i+1}, \ldots, p_{v_*}\},$$

z being an arbitrary point of C^n. Let

(6) $$L^{(i)}(z,p^{(v)}) = \frac{V_i(z,p^{(v)})}{V(p^{(v)})}, \quad i = 1,2,\ldots,v_*.$$

We have

(7) $$L^{(i)}(p_j,p^{(v)}) = \delta_{ij}, \quad i,j = 1,2,\ldots,v_*.$$

The degree of $L^{(i)}(z, p^{(v)})$ is equal to v. We obtain the following:

LEMMA 1. *If* $p^{(v)} = \{p_1, \ldots, p_{v_v}\}$ *is unisolvent system of* v_v *points of* C^n (*i.e.,* $V(p^{(v)}) \neq 0$) *and* $P_v(z)$ *is an arbitrary polynomial of degree* v, *then*

(8) $$P_v(z) = \sum_{i=1}^{v_*} P_v(p_i) L^{(i)}(z,p^{(v)}), \quad z \in C^n.$$

Formula (8) reduces in the case of $n = 1$ to the well-known interpolation formula of Lagrange.

Let $\zeta_k^{(v)} = \{\zeta_{k0}, \zeta_{k1}, \ldots, \zeta_{kv}\}$ be a system of $v + 1$ points of the (z_k)-plane, $k = 1, 2, \ldots, n$, respectively. Let $L^{(i)}(z_k, \zeta_k^{(v)})$ denote the fundamental polynomial (6) of the complex variable z_k corresponding to the system $\zeta_k^{(v)}$. If $P(z) = P(z_1, \ldots, z_n)$ is a polynomial of degree v with respect to any of its variables separately, then by iteration of Lagrange's formula for one variable, we obtain

(9) $$P(z) = \sum_{i_1, i_2, \ldots, i_n = 0}^{v} P(\zeta_{1i_1}, \zeta_{2i_2}, \ldots, \zeta_{ni_n}) L^{(i_1)}(z_1, \zeta_1^{(v)}) \ldots L^{(i_n)}(z_n, \zeta_n^{(v)}).$$

Sometimes it is convenient to have a special interpolation formula for homogeneous polynomials. The simplest way of deriving such a formula is to introduce a system of points unisolvent with respect to homogeneous polynomials. Let $Q_v(z) = Q_v(z_1, \ldots, z_n)$ be a homogeneous polynomial of degree v,

$$Q_v(z) = \sum_{k_1 + \ldots + k_n = v} a_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n},$$

where $\sum_{k_1 + \ldots + k_n = v}$ denotes summation over all solutions in nonnegative integers of the equation $x_1 + x_2 + \ldots + x_n = v$. Let

$$(k_{1l}, k_{2l}, \ldots, k_{nl}), \quad l = 1,2,\ldots,v_0, v_0 = C_{v+n-1,n-1},$$

be a complete sequence of these solutions. Then

(10) $$Q_v(z) = \sum_{l=1}^{v_0} a_{k_{1l} k_{2l} \ldots k_{nl}} z_1^{k_{1l}} \ldots z_n^{k_{nl}}.$$

Let $p^{(v)} = \{p_1, \ldots, p_{v_0}\}$ denote a system of v_0 points of C^n where

$$p_i = (z_{1i}, z_{2i}, \ldots, z_{ni}), \quad i = 1, 2 \ldots, v_0,$$

such that the determinant

$$(11) \qquad W(p^{(v)}) = \det\left[\, z_{1i}^{k_1!}z_{2i}^{k_2!} \cdots z_{ni}^{k_n!}\,\right], \qquad i,l = 1,2,\ldots,v_0,$$

is different from zero. Let $W_i(z, p^{(v)})$ denote the determinant (11) corresponding to the system $\{p_1,\ldots,p_{i-1},\, z,\, p_{i+1},\ldots,p_{v_0}\}$, z being an arbitrary point of C^n. The polynomial

$$(12) \qquad T^{(i)}(z,p^{(v)}) \;=\; \frac{W_i(z,p^{(v)})}{W(p^{(v)})}\,, \qquad i = 1,2,\ldots,v_0,$$

is, of course, homogeneous and of degree v. Moreover

$$(13) \qquad T^{(i)}(p_j,p^{(v)}) \;=\; \delta_{ij}\,, \qquad i,j = 1,2,\ldots,v_0,$$

whence we have.

LEMMA 2. *If* $p^{(v)} = \{p_1,\ldots,p_{v_0}\}$, $v_0 = C_{v+n-1,n-1}$, *is a system of* v_0 *points of* C^n *such that* $W(p^{(v)}) \neq 0$ *and if* $Q_v(z)$ *is an arbitrary homogeneous polynomial of degree* v, *then*

$$(14) \qquad Q_v(z) \;=\; \sum_{i=1}^{v_0} Q_v(p_i)\, T^{(i)}(z,p^{(v)}) \qquad z \in C^n\;.$$

3. Interpolation series of Newton. Let

$$(1) \qquad \zeta_k^{(v)} = \{\zeta_{k0}, \zeta_{k1},\ldots,\zeta_{kv}\}$$

be a system of $v + 1$ distinct points of the z_k-plane, $k = 1,2,\ldots,n$, respectively. Consider the points in C^n given by

$$(2) \qquad p_{i_1 \cdots i_n} = (\zeta_{1i_1}, \zeta_{2i_2},\ldots,\zeta_{ni_n}),$$

where $i_1, i_2,\ldots,i_n$ are nonnegative integers such that

$$(2') \qquad i_1 + i_2 + \ldots + i_n \leqq v.$$

There are $v_* = C_{v+n,n}$ such points. We remember that v_* is also the number of coefficients of a polynomial $P_v(z)$, $z \in C^n$, of degree v.

LEMMA 1. *Given* v_* *arbitrary complex numbers*

$$b_{i_1 i_2 \ldots i_n}\,, \qquad i_1 + \ldots + i_n \leqq v,$$

there exists exactly one polynomial $P_v(z)$ *of degree* v *such that*

$$(3) \qquad P_v(p_{i_1 \ldots i_n}) \;=\; b_{i_1 \ldots i_n}\,, \qquad i_1 + \ldots + i_n \leqq v.$$

Moreover polynomial P_v *may be written in an unique way in the form*

$$(4) \qquad P_v(z) \;=\; \sum_{i_1 + \ldots + i_n \leqq v} a_{i_1 \ldots i_n} \prod_{k=1}^{n} (z_k - \zeta_{k0})\ldots(z_k - \zeta_{ki_k-1}),$$

where by definition we put $(z_k - \zeta_{k,-1}) = 1$.

Proof. At first we shall prove that there is a unique polynomial of the form (4) which satisfies (3). For $n = 1$ the formula (4) reduces to the well-known interpolation formula of Newton. For $n = 2$ see [22]. We shall use induction. Suppose our theorem to be true for $n - 1$ variables and observe that

$$
\begin{aligned}
P_v(z) &= \sum_{i_1=0}^{v} \left[\sum_{i_2+\ldots+i_n \leq v-i_1} a_{i_1 i_2 \ldots i_n} \prod_{k=2} (z_k - \zeta_{k0}) \ldots (z_k - \zeta_{k i_k - 1}) \right] \\
&\quad \times (z_1 - \zeta_{10}) \ldots (z_1 - \zeta_{1 i_1 - 1}) \\
&= \sum_{i_1=0}^{v} P_{v-i_1}(z_2, \ldots, z_n)(z_1 - \zeta_{10}) \ldots (z_1 - \zeta_{1 i_1 - 1}).
\end{aligned}
$$

Equations (3) may now be written in the form

$$
\begin{aligned}
b_{0 i_2 i_3 \ldots i_n} &= P_v(\zeta_{2 i_2}, \zeta_{3 i_3}, \ldots, \zeta_{n i_n}), \qquad i_2 + i_3 + \ldots + i_n \leq v \\
b_{1 i_2 i_3 \ldots i_n} &= P_v(\zeta_{2 i_2}, \zeta_{3 i_3}, \ldots, \zeta_{n i_k}) \\
&\quad + P_{v-1}(\zeta_{2 i_2}, \zeta_{3 i_3}, \ldots, \zeta_{n i_n})(\zeta_{11} - \zeta_{10}), \quad i_2 + \ldots + i_n \leq v-1
\end{aligned}
$$

(4')

$$
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
$$

$$
\begin{aligned}
b_{v 0 0 \ldots 0} &= P_v(\zeta_{20}, \zeta_{30}, \ldots, \zeta_{n0}) + \ldots \\
&\quad + P_0(\zeta_{20}, \ldots, \zeta_{n0})(\zeta_{1v} - \zeta_{10}) \ldots (\zeta_{1v} - \zeta_{1v-1}).
\end{aligned}
$$

Due to the induction assumption the first $C_{v+n-1,n-1}$ equations enable us to find all the coefficients $a_{0 i_2 \ldots i_n}$, $i_2 + \ldots + i_n \leq v$, the next $C_{v+n-2,n-1}$ equations enable us to find all the coefficients $a_{1 i_2 \ldots i_n}$, $i_2 + \ldots + i_n \leq v - 1$, provided $a_{0 i_2 \ldots i_n}$ have been found, and so on. We shall find all the coefficients $a_{i_1 \ldots i_n}$, $i_1 + \ldots + i_n \leq v$, successively. However, these coefficients are uniquely determined by (3). By the way, we see that for any values $b_{i_1 \ldots i_n}$, $i_1 + \ldots + i_n \leq v$, there is at least one polynomial $P_v(z)$ of degree v such that $P_v(p_{i_1 \ldots i_n}) = b_{i_1 \ldots i_n}$. From the theory of linear equations this implies that the determinant (2.4) corresponding to the points (2) is different from zero. Therefore, there is exactly one polynomial $P_v(z)$ of degree v, which satisfies (3).

One may easily check that the determinant Δ of equations (4') has the form

(5)
$$
\begin{aligned}
\Delta &= \delta \cdot (\zeta_{11} - \zeta_{10})^{C_{v+n-2,n+1}} \\
&\quad \times [(\zeta_{12} - \zeta_{10})(\zeta_{12} - \zeta_{11})]^{C_{v+n-3,n-1}}, \ldots [(\zeta_{1v} - \zeta_{10}) \ldots (\zeta_{1v} - \zeta_{1v-1})]^{C_{n-1,n-1}},
\end{aligned}
$$

where δ depends only on the points of $\zeta_k^{(v)}$ for $k = 2, \ldots, v$. After elementary transformations we obtain from (5)

$$
\Delta = \delta \prod_{s=0}^{v-1} [V_s(\zeta_1^{(v)})]^{C_{s+n-2,n-2}},
$$

where

$$
V_s(\zeta_1^{(v)}) = \prod_{0 \leq i < j \leq v-s} (\zeta_{1j} - \zeta_{1i}).
$$

Since the systems $\zeta_k^{(v)}$, $k = 1, 2, \ldots, n$, play an equivalent role in (4'), we have

(6)
$$
|\Delta| = \left| \prod_{k=1}^{n} \prod_{s=0}^{v-1} [V_s(\zeta_k^{(v)})]^{C_{s+n-2,n-2}}, \right|.
$$

By this formula we have $\Delta \neq 0$, whence we may obtain another proof of Lemma 1.

Now we shall find an explicit formula for the coefficients $a_{l_1 \ldots l_n}$. For this purpose let D_k be a domain in the z_k-plane with a smooth boundary C_k oriented positively with respect to D_k. Suppose that the points (1) lie in D_k, $k = 1, 2, \ldots, n$, respectively. Then by the residue theorem

$$
(7) \quad \frac{1}{(2\pi i)^n} \int_{C_1} \cdots \int_{C_n} \frac{P_\nu(\zeta_1, \ldots, \zeta_n) d\zeta_1 \ldots d\zeta_n}{\prod_{k=1}^{n} (\zeta_k - \zeta_{k0}) \ldots (\zeta_k - \zeta_{kl_k})}
$$

$$
= \sum_{i_1 + \ldots + i_n \leq \nu} a_{i_1 \cdots i_n} \frac{1}{(2\pi i)^n} \int_{C_1} \cdots \int_{C_n} \prod_{k=1}^{n} \frac{(\zeta_k - \zeta_{k0}) \ldots (\zeta_k - \zeta_{ki_k - 1})}{(\zeta_k - \zeta_{k0}) \ldots (\zeta_k - \zeta_{kl_k})}
$$

$$
\cdot d\zeta_1 \ldots d\zeta_n = a_{l_1 l_2 \ldots l_n}.
$$

On the other hand, the first integral in (7), also by the residue theorem, is equal to

$$
\sum_{j_1 = 0}^{l_1} \sum_{j_2 = 0}^{l_2} \cdots \sum_{j_n = 0}^{l_n} \frac{P_\nu(\zeta_{1j_1}, \zeta_{2j_2}, \ldots, \zeta_{nj_n})}{\prod_{k=0}^{n} (\zeta_{kj_k} - \zeta_{k0}) \ldots |_{(j_k)} \ldots (\zeta_{kj_k} - \zeta_{kl_k})} ,
$$

whence

$$
(8) \quad a_{l_1 \ldots l_n} = \sum_{j_1 = 0}^{l_1} \cdots \sum_{j = 0}^{l_n} \frac{b_{j_1 \cdots j_n}}{\prod_{k=0}^{n} (\zeta_{kj_k} - \zeta_{k0}) \ldots |_{(j_k)} \ldots (\zeta_{kj_k} - \zeta_{kl_k})}
$$

where $|_{(j_k)}$ means that the factor $(\zeta_{kj_k} - \zeta_{kj_k})$ is omitted.

LEMMA 2. *If the function $f(z)$ is holomorphic in the closure of $D = D_1 \times \ldots \times D_n$ and if*

$$
\zeta_{k0}, \ \zeta_{k1}, \ldots, \ \zeta_{k\nu}, \ldots
$$

is a sequence of different points of D_k, $k = 1, 2, \ldots, n$, respectively, then the series

$$
(9) \quad f \sim \sum_{l=0}^{\infty} \sum_{i_1 + \ldots + i_n = l} a_{i_1 \ldots i_n} \prod_{k=1}^{n} (z_k - \zeta_{k0}) \ldots (z_k - \zeta_{ki_k - 1})
$$

where

$$
(10) \quad a_{i_1 \ldots i} = \frac{1}{(2\pi i)^n} \int_{C_1} \cdots \int_{C_n} \frac{f(\zeta_1, \ldots, \zeta_n) d\zeta_1 \ldots d\zeta_n}{\prod_{k=1}^{n} (\zeta_k - \zeta_{k0}) \ldots (\zeta_k - \zeta_{kl_k})}
$$

converges to $f(z)$ at any point $(\zeta_{1l_1}, \zeta_{2l_2}, \ldots, \zeta_{nl_n})$, $l_1, l_2, \ldots, l_n = 0, 1, 2, \ldots$.

Indeed, the integral on the right hand side of (10) is by the residue theorem equal to the expression on the right hand side of (8), in which $b_{j_1 \ldots j_n}$ has been replaced by $f(\zeta_{1j_1}, \ldots, \zeta_{nj_n})$. This implies that the polynomial

$$
P_\nu(z) = \sum_{l=0}^{\nu} \sum_{i_1 + \ldots + i_n = l} a_{i_1 \ldots i_n} \prod_{k=1}^{n} (z_k - \zeta_{k0}) \ldots (z_k - \zeta_{ki_k - 1}),
$$

being a partial sum of the series (9), takes the value $f(\zeta_{1l_1}, \ldots, \zeta_{nl_n})$ at the point $(\zeta_{1l_1}, \ldots, \zeta_{nl_n})$, provided $\nu \geq l_1 + \ldots + l_n$. Thus Lemma 2 is true.

REMARK. Let us observe that the series (9) differs from the multiple New-ton's series

$$(11) \qquad \sum_{i_1, i_2 \ldots i_n = 0}^{\infty} a_{i_1 \ldots i_n} \prod_{k=1}^{n} (z_k - \zeta_{k0}) \ldots (z_k - \zeta_{k i_k - 1})$$

only by a special method of summation. The series (11) is, of course, also con-vergent to f at the points $(\zeta_{1 l_1}, \ldots, \zeta_{n l_n})$, $l_1, \ldots, l_n = 0, 1, 2, \ldots$.

4. **Unisolvent sets.** We shall say that the set $E \subset C^n$ is unisolvent of order v if there is at least one unisolvent system $p^{(v)} \subset E$, i.e., a system such that $V(p^{(v)}) \neq 0$. If E contains unisolvent systems $p^{(v)}$ of any order $v = 0, 1, \ldots$, we say that E is unisolvent. It is easily shown that a set E unisolvent of order v is also unisolvent of order k, $k = 0, 1, \ldots, v - 1$. To see this it is enough to consider the generalized Laplace's development of $V(p^{(v)})$. In the case of one complex variable the deter-minant $V(p^{(v)})$ given by (2.4) is simply a determinant of Vandermonde of order v and therefore any system of $v_* = v + 1$ different points of C^1 is unisolvent with respect to polynomials in one variable of degree v. In the case of C^1 the homo-geneous polynomials $Q_v(z)$ of degree v has the form $Q_v(z) = a z_1^v$, $a = \text{const}$. Here the problem of unisolvency is trivial.

In the space C^n, $n \geq 2$, the unisolvent systems are not so simply characterized. There are systems $p^{(v)} = \{p_1, \ldots, p_{v_*}\}$ and $q^{(v)} = \{q_1, \ldots, q_{v_0}\}$ of different points $\in C^n$ such that $V(p^{(v)}) = 0$ and $W(q^{(v)}) = 0$, respectively.

It follows from Lemma 1, §3, that if E is a set of $(v + 1)^n$ points

$$(1) \qquad p_{i_1 \ldots i_n} = (\zeta_{1 i_1}, \ldots, \zeta_{n i_n}), \qquad 0 \leq i_1, i_2, \ldots, i_n \leq v,$$

where $\zeta_{k0}, \zeta_{k1}, \ldots, \zeta_{kv}$ are different, then the system of points (1) which satisfies $i_1 + \ldots + i_n \leq v$ is unisolvent. In fact, there are at least $[(v + 1)!]^n$ different unisolvent systems of order v in the set E.

COROLLARY 1. *If E contains the Cartesian product of the sequences*

$$\zeta_{k0}, \zeta_{k1}, \ldots, \qquad k = 1, 2, \ldots, n; \qquad (\zeta_{ki} \neq \zeta_{kj} \quad for \ i \neq j),$$

then E is unisolvent with respect to polynomials of n complex variables.

We shall now find the absolute value of the determinant (2.4) which corresponds to the system $p^{(v)}$ of points (1). We denote this determinant by

$$(2) \qquad V((\zeta_{1 i_1}, \ldots, \zeta_{n i_n}), \qquad i_1 + \ldots + i_n \leq v).$$

Let $P_v(z)$ be a polynomial of degree v such that

$$(3) \qquad P_v(p_{i_1 \ldots i_n}) = b_{i_1 \ldots i_n}, \qquad i_1 + \ldots + i_n \leq v,$$

the $b_{i_1 \ldots i_n}$ being arbitrary fixed complex numbers. The determinant of the linear equations (3) with unknowns $a_{i_1 \ldots i_n}$ is equal to (2). The polynomial $P_v(z)$ may be written in the form

$$P_v(z) = \sum_{k_1=0}^{v} \left(\sum_{k_2+\ldots+k_n \leqq v-k_1} a_{k_1 \cdots k_n} z_2 {}_{k_2 \cdots} z_n^{k_n} \right) z_1^{k_1}$$

$$= \sum_{k_1=0}^{v} P_{v-k_1}(z_2, \ldots, z_n) z_1^{k_1},$$

where $P_{v-k_1}(z_2, \ldots, z_n)$ is a polynomial of degree $v - k_1$ of the $n-1$ variables $z_2, \ldots, z_n$. To begin with, let us assume that $\zeta_{10} = 0$. Then the equations (3) may be written in the form

$$P_v(\zeta_{2i_2}, \ldots, \zeta_{ni_n}) = b_{0i_2 \ldots i_n}, \qquad i_2 + \ldots + i_n \leqq v$$

(3′)

$$\zeta_{1i} \sum_{k=0}^{v-1} P_{v-k-1}(\zeta_{2i_2}, \ldots, \zeta_{ni_n}) \zeta_{1i}^k = - P_v(\zeta_{2i_2}, \ldots, \zeta_{ni_n}) + b_{ii_2 \cdots i_n},$$

$$i_2 + \ldots + i_n \leqq v - i, \quad i = 1, 2, \ldots, v$$

whence the following recurrential formula follows for

$$V(p^{(v)}) = {}^{df}V((\zeta_{1i_1}, \ldots, \zeta_{ni_n}), \quad i_1 + \ldots + i_n \leqq v)$$

(4) $V((\zeta_{1i_1}, \ldots, \zeta_{ni_n}), i_1 + \ldots + i_n \leqq v) = V((\zeta_{2i_2}, \ldots, \zeta_{ni_n}), i_2 + \ldots + i_n \leqq v)$

$$\times \; V((\zeta_{1i_1+1}, \zeta_{2i_2}, \ldots, \zeta_{ni_n}), i_1 + \ldots + i_n \leqq v - 1) \zeta_{11}^{C_{v+n-2,n-1}} \zeta_{12}^{C_{v+n-3,n-1}}$$

$$\ldots \zeta_{1v}{}^{C_{n-1,n-1}}.$$

Since the determinant $V(p^{(v)})$ does not depend on unitary transformation of C^n onto itself, it follows that if $\zeta_{10} \neq 0$ then

$$V(p^{(v)}) = V((\zeta_{2i_2}, \ldots, \zeta_{ni_n}), \quad i_2 + \ldots + i_n \leqq v)$$

$$\times \; V((\zeta_{1i_1+1} - \zeta_{10}, \quad \zeta_{2i_2}, \ldots, \zeta_{ni_n}), i_1 + \ldots + i_n \leqq v - 1)$$

(5)

$$\times \; (\zeta_{11} - \zeta_{10})^{C_{v+n-2,n-1}} (\zeta_{12} - \zeta_{10})^{C_{v+n-3,n-1}}$$

$$\ldots (\zeta_{1v} - \zeta_{10})^{C_{n-1,n-1}}.$$

Since $|V(p^{(v)})|$ is symmetric with respect to $\zeta_k^{(v)}$, $k = 0, 1, 2, \ldots, v$, we obtain from (5)

(6) $$\left| V((\zeta_{1i_1}, \ldots, \zeta_{ni_n}), i_1 + \ldots + i_n \leqq v) \right| = \left| \prod_{k=1}^{n} \prod_{s=0}^{v-1} [V_s(\zeta_k^{(v)})]^{C_{s+n-2,n-2}} \right|,$$

where

$$V_s(\zeta_k^{(v)}) = \prod_{0 \leqq , < j \leqq v-s} (\zeta_{kj} - \zeta_{ki}), \qquad s = 0, 1, \ldots, v-1.$$

By the way, we have proved that the absolute value of the determinant of the equations (3) is equal to the absolute value of the determinant of the equations (3.4′).

Let $Q(z) = Q(z_1, \ldots, z_n)$ be a homogeneous polynomial of degree v. Then the function

$$P(z_2, z_3, ..., z_n) = (z_1^0)^v \cdot Q\left(1, \frac{z_2}{z_1^0}, ..., \frac{z_n}{z_1^0}\right), \; z_1^0 = \text{const} \neq 0$$

is an ordinary polynomial in $n - 1$ variables of degree v.

COROLLARY 2. *A set* $E \subset C^n$, *which is a Cartesian product of a point* $z_1^0 \neq 0$ *and of systems* $\zeta_k^{(v)} = \{z_{k0}, z_{k_1}, ..., z_{kv}\}$, $k = 2, ..., n$, *of different points of the* z_k-*plane, is unisolvent of order* v *with respect to homogeneous polynomials.*

5. **Extremal points.** Let E be a bounded closed set in C^n. Let $b(z)$ be a real function defined and bounded on E. Given an arbitrary system $p^{(v)} = \{p_1, ..., p_{v_*}\} \subset E$ of points

$$p_l = (z_{1l}, ..., z_{nl}) \; , \quad l = 1, 2, ..., v_* \; , \quad v_* = C_{v+n,n}$$

we define $V(p^{(v)}, b)$ by

(1) $$V(p^{(v)}, b) = V(p^{(v)}) \exp\left[-v \sum_{l=1}^{v_*} b(p_l)\right], \quad v = 0, 1, 2, ...,$$

where $V(p^{(v)})$ is given by (2.4). Let $\{\alpha_v\}$ be a sequence of real numbers such that

(2) $$\alpha_v > 1 \; , \quad v = 0, 1, ..., \quad \text{and} \quad \lim_{v \to \infty} (\alpha_v)^{1/v} = 1.$$

For any $v = 0, 1, ...$ there is a system

(3) $$\gamma^{(v)} = \{\gamma_1^{(v)}, \; \gamma_2^{(v)}, ..., \gamma_{v_*}^{(v)}\}$$

of points of E such that

(4) $$|V(\gamma^{(v)}, b)| > \alpha_v^{-1} |V(p^{(v)}, b)|, \quad v = 1, 2, ...,$$

$p^{(v)}$ being an arbitrary system of v_* points of E.

The system (3) will be called the vth extremal system of E with respect to $b(z)$, $\{\alpha_v\}$ and $V(p^{(v)})$. The points of system (3) will be called the extremal points of order v.

If $b(z)$ is lower semicontinuous then $V(p^{(v)}, b)$ is upper semicontinuous with respect to $p^{(v)}$. Therefore, in that case there is a system

(3') $$q^{(v)} = \{q_1^{(v)}, \; q_2^{(v)}, ..., q_{v_*}^{(v)}\}$$

of points E such that

(4') $$|V(q^{(v)}, b)| = \max_{p^{(v)} \subset E} |V(p^{(v)}, b)|, \quad v = 0, 1, 2, ... \; .$$

The points (3') will be called ordinary extremal points of E with respect to $b(z)$ (and $V(p^{(v)})$). In the case that $E \subset C^1$ and $b(z) \equiv 0$ the extremal points were introduced by Fekete [5]. In the case that $E \subset C'$ and $b(z)$ is bounded, extremal points were introduced by Leja [11] and investigated later by him and his students in connection with the Dirichlet boundary value problem and conformal mapping of simple and multiconnected domains on some canonical domains (for bibliography see [14]).

Using the same procedure, one may introduce extremal points of E connected with homogeneous polynomials. If $b(z)$ is lower semicontinuous, there exists a system

(5) $$h^{(v)} = \{h_1^{(v)},\ h_2^{(v)}, ..., h_{v_0}^{(v)}\}, \qquad v_0 = C_{v+n-1,n-1},$$

of points of E such that

(6) $$\left|W(h^{(v)},b)\right| = \max_{p^{(v)} \subset E} \left|W(p^{(v)},b)\right|, \qquad p^{(v)} = \{p_1, ..., p_{v_0}\},$$

where

$$W(p^{(v)},b) = W(p^{(v)})\exp\left[-v \sum_{i=1}^{v_0} b(p_i) \right], \qquad v = 0,1,2, ... \ .$$

In the case of $n = 2$ the extremal points (5) have been introduced by Leja [9] ($b(z) \equiv 0$) and applied by him to the investigation of domains of uniform convergency of the series of homogeneous polynomials of two complex variables (see [9;12]). Exploitation of the points (4) ($b \equiv 0$) to the same purpose in the case of C^n, $n \geq 3$, has been done in [19].

Let us define $v_v(E,b)$ and $w_v(E,b)$ by

(7) $$v_v(E,b) = \left[\left|V(q^{(v)},b)\right| \right]^{1/n\, C_{v+n,n+1}}, \ v = 1,2,...,$$

(8) $$w_v(E,b) = \left[\left|W(h^{(v)},b)\right| \right]^{1/v\, C_{v+n-1,n-1}} , \quad v = 1,2,.... \ .$$

One can prove that the numbers $v_v(E,0)$ and $w_v(E,0)$ are invariant with respect to the unitary transformations of C^n onto itself.

It is known [9;10;11;14] that the sequence $\{v_v(E,b)\}$, $E \subset C^1$, and the sequence $\{w_v(E,b)\}$, $E \subset C^2$, are both convergent. Convergence of $\{w_v(E,b)\}$ for $E \subset C^1$ is trivial. The limit $v(E,0) = \lim v_v(E,0)$ is called the transfinite diameter of E (= logarithmic capacity of E). The limit $w(E,0) = \lim w_v(E,0)$, $E \subset C^2$, is a triangular transfinite diameter of E [9;12]. The question (formulated by Leja [16] in a slightly different form) as to whether the sequences (7) or (8) for $E \subset C^n$, $n \geq 2$, are convergent or not remains still unsolved (except for $E = E_1 \times E_2 \times ... \times E_n$).

Remark on Šilov's boundary. Let $S(E)$ denote Šilov's boundary of E with respect to polynomials and let E^* denote the topological sum of all ordinary extremal points of E with respect to $b(z) \equiv 0$. The extremal points of $q^{(v)}$ are not unique in general. Therefore, E^* may a priori depend on which extremal points of order v we choose for $v = 1,2,...$. There is, of course, at least one E^* such that $\bar{E}^* \subset S(E)$. But, as we shall see from Lemma 1, §6, any polynomial takes its maximum on $\bar{E}^*$. Therefore, we always have $S(E) \subset \bar{E}^*$. We know [18] that in the case of C^1 the set E^* is unique and therefore $\bar{E}^* = S(E)$. The author can prove that $\bar{E}^*$ is unique also if $E \subset C^n$, $n = 2$, is circular. However, we do not know what is the answer in the general case.

6. **The extremal function** $\Phi(z,E,b)$. Let E be a bounded closed unisolvent set in C^n. Given an arbitrary real function $b(z)$ defined and bounded on E and an arbitrary unisolvent system

$$p^{(v)} = \{p_1, ..., p_{v_*}\}, \qquad v = 1,2,...,v_* = C_{v+n,n},$$

of points of E, the functions

(1) $$\Phi^{(i)}(z,p^{(v)},b) = L^{(i)}(z,p^{(v)})e^{vb(p_i)}, \qquad i = 1,2, ..., v_*,$$

where $L^{(i)}(z,p^{(v)})$ denote the polynomials (2.6), are polynomials of degree v such that

(2) $$\Phi^{(i)}(p_j,p^{(v)}, b) = \delta_{ij}e^{vb(p_i)}, \qquad i,j = 1,2,...,v_*.$$

For any $v = 1,2,...$ we define extremal functions $\Phi_v^{(i)}(z,E,b)$, $i = 1,2,3,4$, corresponding to E and b, by the formulas

(3) $$\Phi_v^{(1)}(z,E,b) = \max_{1 \leq i \leq v_*} \left| \Phi^{(i)}(z,\gamma^{(v)},b) \right|,$$

(4) $$\Phi_v^{(2)}(z,E,b) = \sum_{i=0}^{v_*} \left| \Phi^{(i)}(z,\gamma^{(v)},b) \right|,$$

(5) $$\Phi_v^{(3)}(z,E,b) = \inf_{p^{(v)} \subset E} \{ \max_{(i)} \left| \Phi^{(i)}(z,p^{(v)},b) \right| \},$$

(6) $$\Phi_v^{(4)}(z,E,b) = \inf_{p^{(v)} \subset E} \sum_{i=1}^{v_*} \left| \Phi^{(i)}(z,p^{(v)},b) \right|,$$

where $\gamma^{(v)} = \{\gamma_1^{(v)}, ..., \gamma_{v_*}^{(v)}\}$ denotes extremal system (5.3).

THEOREM 1. *The sequences* $\{[\Phi_v^{(i)}(z,E,b)]^{1/v}\}$, $i = 1,2,3,4$, *are convergent at any point* $z \in C^n$ *to the same limit* $\Phi(z,E,b)$,

$$\Phi(z,E,b) = \lim_{v \to \infty} [\Phi_v^{(i)}(z,E,b)]^{1/v}, \qquad z \in C^n , \quad i = 1,2,3,4,$$

(*the limit* $\Phi(z,E,b)$ *being finite or not*).

Proof. 1° First of all we shall prove that the sequence $\{(\Phi_{(v)}^{(1)})^{1/v}\}$ has a limit (finite or not) at any point $z \in C^n$. Due to (2.6) and (5.1) we have

$$\Phi^{(i)}(z,\gamma^{(v)},b) = \frac{V_i(z,\gamma^{(v)},b)}{V(\gamma^{(v)}, b)} , \qquad i = 1,2 ...,v_*,$$

where $V_i(z,\gamma^{(v)},b)$ is a determinant (5.1) corresponding to $\{\gamma_1^{(v)}, \gamma_2^{(v)},...,\gamma_{i-1}^{(v)}, z, \gamma_{i+1}^{(v)},...,\gamma_{v_*}^{(v)}\}$. Therefore, in virtue of (5.4)

(7) $$\left| \Phi^{(i)}(z,\gamma^{(v)},b) \right| < \alpha_v \exp[vb(z)], \qquad z \in E, \quad i = 1,2, ...,v_*.$$

Let z be an arbitrary fixed point of C^n, let v be an arbitrary fixed positive integer and let μ be an arbitrary integer greater or equal to v. There exist two

uniquely determined integers k and r such that $\mu = k\nu + r$ and $0 \leqq r < \nu$. By the interpolation formula (2.8) and due to inequality (7) we have

$$\left| [\Phi^{(i)}(z,\gamma^{(\nu)},b)]^k \right| \leqq \alpha_\nu^k \sum_{j=1}^{\mu_*} \exp\left[k\nu b(\gamma_j^{(\mu)})\right] \left| L^{(j)}(z,\gamma^{(\mu)}) \right|, \qquad i = 1,2,\ldots,\nu_*.$$

Let $b_0 = \inf_{z\in E} b(z)$. Then

$$\exp\left[k\nu b(\gamma_j)\right] = \exp\left[(\mu - r)b(\gamma_j)\right] \leqq \exp\left[\mu\, b(\gamma_j) - rb_0\right].$$

Therefore,

$$\left|\Phi^{(i)}(z,\gamma^{(\nu)},b)\right|^k \leqq \alpha_\nu^k \exp[-rb_0] \sum_{j=1}^{\mu_*} \left|\Phi^{(j)}(z,\gamma^{(\mu)},b)\right| \leqq \mu_* \alpha_\nu^k \exp[-r\,b_0]\, \Phi_\mu^{(1)}(z,E,b),$$

whence

$$\left[(\Phi_\nu^{(1)}(z,E,b))^{1/\nu}\right]^{\nu k/\mu} \leqq (\alpha_\nu)^{1/\nu}\left[\mu_*\, e^{-rb_0}\right]^{1/\mu}\left[\Phi_\mu^{(1)}(z,E,b)\right]^{1/\mu}, \qquad \mu = 1,2,\ldots.$$

Since $\nu k/\mu \to 1$ and $(\mu_* \exp(-r\,b_0))^{1/\mu} \to 1$, as $\mu \to \infty$, we have

$$(8) \qquad \left[\Phi_\nu^{(1)}(z,E,b)\right]^{1/\nu} \leqq (\alpha_\nu)^{1/\nu} \liminf_{\mu\to\infty}\ \left[\Phi_\mu^{(1)}(z,E,b)\right]^{1/\mu}, \qquad \nu = 1,2,\ldots,$$

whence due to (5.2)

$$\limsup_{\nu\to\infty}\ \left[\Phi_\nu^{(1)}(z,E,b)\right]^{1/\nu} \leqq \liminf_{\mu\to\infty}\ \left[\Phi_\mu^{(1)}(z,E,b)\right]^{1/\mu}, \qquad z \in C^n.$$

Therefore, the sequence $\{(\Phi_\nu^{(1)})^{1/\nu}\}$ has the limit $\Phi(z) = \Phi(z,E,b)$ (finite or not).

2° To prove that the sequences $\{[\Phi^{(i)}(z,E,b)]^{1/\nu}\}$, $i = 2,3,4$, are convergent to $\Phi(z)$ it is sufficient to show that

$$(9) \qquad \Phi_\nu^{(1)} \leqq \alpha_\nu \Phi_\nu^{(4)} \leqq \alpha_\nu \Phi_\nu^{(2)} \leqq (\nu_*)^2 \alpha_\nu^2 \Phi_\nu^{(3)} \leqq (\nu_*)^2 \alpha_\nu^2 \Phi_\nu^{(1)}, \qquad \nu = 1,2,\ldots\ .$$

By the interpolation formula (2.8) and due to (7) we have

$$(10) \qquad \left|\Phi^{(i)}(z,\gamma^{(\nu)},b)\right| \leqq \alpha_\nu \sum_{j=1}^{\nu_*} \left|\Phi^{(j)}(z,p^{(\nu)},b)\right|, \qquad z \in C^n, \quad i = 1,2,\ldots,\nu_*,$$

$p^{(\nu)}$ being an arbitrary unisolvent system in E of order ν. Thus $\Phi_\nu^{(1)} \leqq \alpha_\nu \Phi_\nu^{(4)}$. The inequality $\Phi_\nu^{(4)} \leqq \Phi_\nu^{(2)}$ follows directly from (4) and (6). Further, it follows from (10) that

$$\sum_{i=1}^{\nu_*} \left|\Phi^{(i)}(z,\gamma^{(\nu)},b)\right| \leqq \nu_*^2 \alpha_\nu \inf_{p^{(\nu)}\subset E} \{\max_{(j)} |\Phi^{(j)}(z,p^{(\nu)},\,b)|\},$$

whence $\Phi_\nu^{(2)} \leqq \nu_*^2 \alpha_\nu \Phi_\nu^{(3)}$. Finally, the inequality $\Phi_\nu^{(3)} \leqq \Phi_\nu^{(1)}$ follows directly from (3) and (5). The proof is completed.

LEMMA 1. *If $P_\nu(z)$ denotes an arbitrary polynomial of degree ν such that*

$$(11) \qquad \left|P_\nu(z)\right| \leqq M\, e^{\nu b(z)}, \qquad z \in E, \quad M = \text{const},$$

then

$$(12) \qquad \left|P_\nu(z)\right| \leqq M\, \Phi^\nu(z,E,b), \qquad z \in C^n.$$

Proof. By the interpolation formula (2.8) and due to (11) we have

$$\left|[P_\nu(z)]^\mu\right| \le M^\mu \sum_{i=1}^{(\mu\nu)_*} \left|\Phi^{(i)}(z,\gamma^{(\mu\nu)},b)\right| = M^\mu \Phi_{\mu\nu}^{(2)}, \qquad \mu = 1,2,\dots .$$

Therefore, $\left|P_\nu(z)\right| \le M[\Phi_{\mu\nu}^{(2)}(z,E,b)]^{1/\mu}$, $\mu = 1,2,\dots$, whence (12) follows.

Let $D(E)$ denote the unbounded component of CE. We know [10;7] that if $E \subset C^1$ is of positive logarithmic capacity and $b(z) \equiv 0$, then $\Phi(z,E,0) \equiv \exp G(z,E)$, where $G(z,E)$ is a Green's function of $D(E)$ with pole at ∞. Therefore, (12) is a generalization of the Bernstein-Walsh inequality [27, p. 77].

THEOREM 2. *Let* $A_\nu(E,b)$ *denote the family of all polynomials of degree* ν *such that* $\left|P_\nu(z)\right| \le M_\nu \exp[\nu b(z)]$, $z \in E$, $M_\nu = \text{const}$, $(M_\nu)^{1/\nu} \to 1$. *Then*

$$(13) \qquad \Phi(z,E,b) = \lim_{\nu \to \infty} \left\{ \sup_{P_\nu \in A_\nu} (\left|P_\nu(z)\right|)^{1/\nu} \right\}, \quad z \in C^n.$$

Proof. By Lemma 1 we have

$$\left|P_\nu(z)\right| \le M_\nu \Phi^\nu(z,E,b), \quad z \in C^n, \quad \nu = 1,2,\dots.$$

On the other hand, by (7) the polynomials $\alpha_\nu^{-1} M_\nu \Phi^{(i)}(z,\gamma^{(\nu)},b)$, $i = 1,2,\dots,\nu_*$, belong to $A_\nu(E,b)$, $\nu = 1,2,\dots$. Therefore (13) is true.

REMARK. Let

$$R_\nu(z,E,b) = \sup_{P_\nu \in A_\nu} \left|\frac{P_\nu(z)}{M_\nu}\right|, \qquad \nu = 1,2\dots.$$

Then $R_{\mu+\nu}(z) \ge R_\mu(z)R_\nu(z)$, $\mu,\nu = 1,2,\dots$, whence it follows that there exists the limit $R(z) = \lim_{\nu \to \infty} [R_\nu(z,E,b)]^{1/\nu}$, $z \in C^n$.

7. **Some fundamental properties of** $\Phi(z,E,b)$. Let $b_0 = \inf_{z \in E} b(z)$ and $B_0 = \sup_{z \in E} b(z)$ Then

$$(1) \qquad e^{b_0} \le e^{b_0}\Phi(z,E,0) \le \Phi(z,E,b) \le e^{B_0}\Phi(z,E,0), \qquad z \in C^n .$$

Indeed, since $\left|\Phi^{(i)}(z,\gamma^{(\nu)},b)\right| \ge \left|L^{(i)}(z,\gamma^{(\nu)})\right|e^{\nu b_0}$, $i = 1,2,\dots,\nu_*$, then $\Phi_\nu^{(1)}(z,E,b) \ge \Phi_\nu^{(3)}(z,E,0)e^{\nu b_0}$, whence $\Phi(z,E,b) \ge \Phi(z,E,0)e^{b_0}$. On the other hand,

$$\sum_{i=1}^{\nu_*} L^{(i)}(z,\gamma^{(\nu)}) \equiv 1, \; z \in C^n,$$

therefore, $\Phi_\nu^{(1)}(z,E,0) \ge 1/\nu_*$, whence

$$(2) \qquad \Phi(z,E,0) \ge 1, \qquad z \in C^n.$$

Thus we proved the first two inequalities in (1).

Since for any unisolvent system $p^{(\nu)} \subset E$ we have

$$\left|\Phi^{(i)}(z,p^{(\nu)},b)\right| < \exp(\nu B_0)\left|L^{(i)}(z,p^{(\nu)})\right|,$$

then $\Phi_\nu^{(4)}(z,E,b) \le \exp(\nu B_0) \Phi_\nu^{(4)}(z,E,0)$, whence the last inequality of (1) follows.

The complement CE of E in the space C^n consists of at most a countable num-

ber of disjoint domains $CE = \{D_\nu\} + D_\infty$, where $D_\infty = D(E)$ is unbounded. By the maximum modulus principle for holomorphic functions the ordinary extremal points of E with respect to $b(z) \equiv 0$ lie on the boundary of D_∞. Therefore the extremal systems of E and $\Delta = U\bar{D}_\nu$ with respect to $b(z) \equiv 0$ are the same. Thus,

$$(3) \qquad\qquad \Phi(z, E, 0) \equiv \Phi(z, \Delta, 0), \qquad z \in C^n.$$

As a simple consequence of (6.7), (2) and (3) we obtain

$$(4) \qquad\qquad \Phi(z, E, 0) = 1 \qquad \text{for } z \in \Delta.$$

The following three properties follow directly from the definition (6.6) of $\Phi_\nu^{(4)}$ and from Theorem 1, §6:

$$(5) \quad \Phi(z, E, b_1) = e^c \Phi(z, E, b), \qquad z \in C^n, \qquad b_1(z) = b(z) + c, \qquad c = \text{const};$$

$$(6) \qquad\qquad \Phi(z, E, b) \leqq \Phi(z, F, b),$$

$z \in C^n$, if $F \subset E$ is a unisolvent closed subset of E; and

$$(7) \qquad\qquad \Phi(z, E, b_1) \leqq \Phi(z, E, b_2),$$

$z \in C^n$, if $b_1(z) \leqq b_2(z)$ for $z \in E$.

Now we shall prove a less obvious property of Φ, namely,

$$(8) \qquad \prod_{i=1}^s \Phi(z, E, b_i) \leqq \Phi^s(z, E, b), \qquad z \in C^n, \qquad b(z) = \frac{1}{s}[b_1(z) + \dots + b_s(z)].$$

We shall prove (8) for $s = 2$. If $s > 2$, the proof is quite analogous. Let $b(z) = [b_1(z) + b_2(z)]/2$ and let $\gamma^{(\nu,i)} = \{\gamma^{(\nu,i)}, \dots, \gamma_{\nu_*}^{(\nu,i)}\}$, $i = 1,2$, be the νth extremal system of E with respect to $b_i(z)$, $i = 1, 2$, respectively. Let

$$\gamma^{(2\nu)} = \{\gamma_1^{(2\nu)}, \gamma_2^{(2\nu)}, \dots, \gamma_{(2\nu)_*}^{(2\nu)}\}$$

be the (2ν)th extremal system of E with respect to $b(z)$. Given $z_0 \in C^n$, there are integers i_1 and i_2 such that

$$\left| \Phi^{(i_k)}(z_0, \gamma^{(\nu,k)}, b_k) \right| = \Phi_\nu^{(1)}(z_0, E, b_k), \qquad k = 1,2.$$

Due to the interpolation formula (2.8) and by (6.7) we have

$$\left| \Phi^{(i_1)}(z, \gamma^{(\nu,1)}, b_1) \Phi^{(i_2)}(z, \gamma^{(\nu,2)}, b_2) \right|$$

$$\leqq \alpha_\nu^2 \sum_{i=1}^{(2\nu)_*} \exp\left(2\nu \frac{b_1(\gamma_i^{(2\nu)}) + b_2(\gamma_i^{(2\nu)})}{2} \left| L^{(i)}(z, \gamma^{(2\nu)}) \right| \cdot \right)$$

Therefore,

$$\Phi_\nu^{(1)}(z_0, E, b_1) \Phi_\nu^{(1)}(z_0, E, b_2) \leqq \alpha_\nu^2 \Phi_{2\nu}^{(2)}(z_0, E, b), \qquad \nu = 1, 2, \dots,$$

whence (10) follows.

Due to (6.8) and Theorem 1, §6, we have

$$(9) \qquad\qquad [\Phi_\nu^{(1)}(z, E, b)]^{1/\nu} \leqq \Phi(z, E, b), \qquad z \in C^n, \quad \nu = 1, 2, \dots.$$

Thus Φ is an upper bound of continuous functions $(\Phi_\nu^{(1)})^{1/\nu}$, $\nu = 1, 2, \ldots$. Therefore, $\Phi(z, E, b)$ is lower semicontinuous in C^n. Function $(1/\nu) \log \left| \Phi^{(i)}(z, \gamma^{(\nu)}, b) \right|$, $i = 1, 2, \ldots, \nu_*$, is plurisubharmonic in C^n, so the function $(1/\nu) \log \Phi_\nu^{(i)}(z, E, b) = \max_i (1/\nu) \log \left| \Phi^{(i)}(z, \gamma^{(\nu)}, b) \right|$ is also plurisubharmonic in C^n. Since $\Phi(z, E, b)$ is an upper limit of plurisubharmonic functions $[\Phi_\nu^{(1)}(z, E, b)]^{1/\nu}$, the function

$$(10) \qquad \log \Phi^*(z, E, b) = \lim_{z' \to z} \sup \log \Phi(z', E, b), \qquad z, \, z' \in C^n,$$

is also plurisubharmonic in any domain $D \subset C^n$ in which $\Phi(z, E, b)$ is bounded.

COROLLARY 1. *If E is a Šilov boundary (with respect to polynomials) of a bounded domain D, then $\log \Phi^*(z, E, b)$ is plurisubharmonic in D.*

Let $\mathscr{P}(E)$ denote a set of the points $z_0 \in C^n$ such that for any polynomial $P(z)$ we have

$$\left| P(z_0) \right| \leq \max_{z \in E} \left| P(z) \right|.$$

We claim that

$$(11) \qquad \mathscr{P}(E) = \{ z \mid \Phi(z, E, 0) = 1 \} .$$

At first we shall prove that if $z_0 \in \mathscr{P}(E)$, then $\Phi(z_0, E, 0) = 1$. If this was not true then by (2) we would have $\Phi(z_0, E, 0) > 1$. Therefore, there would exist integers ν and i_0, $1 \leq i_0 \leq \nu_*$, such that $\left| L^{(i_0)}(z_0, \gamma^{(\nu)}) \right| > 1$. But $L^{(i_0)}(z, \gamma^{(\nu)})$ is a polynomial of degree ν and $\left| L^{(i_0)}(z, \gamma^{(\nu)}) \right| \leq 1$ for $z \in E$, whence by the definition of $\mathscr{P}(E)$ we would have $\left| L^{(i_0)}(z_0, \gamma^{(\nu)}) \right| \leq 1$. We have obtained a contradiction. Thus $\Phi(z_0, E, 0) = 1$. On the other hand, if $\Phi(z_0, E, 0) = 1$ and $P(z)$ is an arbitrary polynomial of degree ν, then due to (6.12) we have

$$\left| P(z_0) \right| \leq (\max_{z \in E} \left| P(z) \right|) \Phi^\nu(z_0, E, 0) = \max_{z \in E} \left| P(z) \right|.$$

Therefore, if $\Phi(z_0, E, 0) = 1$ then $z_0 \in \mathscr{P}(E)$. The proof is completed.

8. **The function** $\Phi(z, E, 0)$ **for** $E = E_1 \times E_2 \times \ldots \times E_n$. The following lemma has been proved in [15].

LEMMA 1. *If E is a compact plane set of positive logarithmic capacity, then there exist points $x_\nu \in E$, $\nu = 0, 1, \ldots$, such that $x_i \neq x_j$ for $i \neq j$ and*

$$(1) \qquad \left| \omega_\nu(z) \right| \leq \left| \omega_\nu(x_\nu) \right|, \qquad z \in E, \quad \nu = 1, 2, \ldots,$$

where

$$(2) \qquad \omega_\nu(z) = (z - x_0)(z - x_1) \ldots (z - x_{\nu-1}), \qquad \nu = 1, 2, \ldots.$$

Moreover, the sequence $\{ (\left| \omega_\nu(z)/\omega_\nu(x_\nu) \right|)^{1/\nu} \}$ converges uniformly to $\Phi(z, E, 0)$ on any closed subset of CE.

Now we shall prove

LEMMA 2. *Let $E = E_1 \times \ldots \times E_n$, where E_k is a compact set of positive logarithmic capacity in the complex z_k-plane. If $P_\nu(z) = P_\nu(z_1, \ldots, z_n)$ is a polynomial of degree ν such that*

(3) $\left| P_v(z) \right| \leqq M \quad for \quad z \in E,$

then

(4) $\left| P_v(z) \right| \leqq M \max \{\Phi^v(z_1, E_1, 0), ..., \Phi^v(z_n, E_n, 0)\}, \qquad z \in C^n.$

Proof. Let

$$x_{k0}, \; x_{k1}, ..., \; x_{kv}, ...$$

denote the sequence of points of E_k whose existence is assured by Lemma 1. Let

$$\omega_{kv}(z_k) = (z_k - x_{k0}) ... (z_k - x_{kv-1}), \qquad v = 1, 2, ..., \qquad k = 1, 2, ..., n.$$

Then

(5) $\left| \dfrac{\omega_{kv}(z_k)}{\omega_{kv}(x_{kv})} \right| \leqq 1, \qquad z_k \in E_k, \quad v = 1, 2, ...,$

and the sequence $\{(\left| \omega_{kv}(z_k) / \omega_{kv}(x_{kv}) \right|)^{1/v}\}$ converges uniformly to $\Phi(z_k, E_k, 0)$ on any closed subset of CE_k. By Newton's interpolation formula we have

(6) $P_v^\mu(z) \equiv \sum\limits_{i_1 + ... + i_n \leqq \mu v} a_{i_1 i_2 ... i_n} \prod\limits_{k=1}^{n} \omega_{k i_k}(z_k), \quad (\omega_{k0}(z_k) \equiv^{df} 1),$

where

(7) $a_{i_1 ... i_n} = \dfrac{1}{(2\pi i)^n} \int\limits_{C_1} ... \int\limits_{C_n} \dfrac{P_v^\mu(\zeta_1, ..., \zeta_n) d\zeta_1 ... d\zeta_n}{\prod\limits_{k=1}^{n} \omega_{k i_k + 1}(\zeta_k)},$

$C_1, ..., C_n$ being smooth suitably oriented curves, which contain in their interiors $E_1, E_2, ..., E_n$, respectively. Since E_k may be approximated from the outside by regular sets (for which the function Φ is continuous) and the function $\Phi(z_k, E_k, 0)$ is continuous with respect to the sets, it is sufficient to prove our lemma only for regular sets. Assuming E_k to be regular and ε to be an arbitrary positive number, let $R > 1$ be so near to 1 that $R - \varepsilon < 1$ and

$$\left| P_v(z) \right| < (1 + \varepsilon)M \quad for \quad z \in C = C_1 \times C_2 \times ... \times C_n,$$

where

$$C_k = \{z_k \,|\, \Phi(z_k, E_k, 0) = R\}, \qquad k = 1, 2, ..., n.$$

For these C_k we have by (3) and (7)

$$\left| a_{i_1 ... i_n} \right| \leqq (1 + \varepsilon)^\mu M^\mu M_1 \Big/ \prod\limits_{k=1}^{n} \min\limits_{\zeta_k \in C_k} \left| \omega_{k i_k}(\zeta_k) \right|,$$

M_1 being a constant which depends only on C. Therefore,

$$\left| P_v^\mu(z) \right| \leqq (1 + \varepsilon)^\mu M^\mu M_1 \sum\limits_{i_1 + ... + i_n \leqq uv} \prod\limits_{k=1}^{n} \left| \dfrac{\omega_{k i_k}(z_k)}{\omega_{k i_k}(x_{k i_k})} \right| \prod\limits_{k=1}^{n} \max\limits_{\zeta_k \in C_k} \left| \dfrac{\omega_{k i_k}(x_{k i_k})}{\omega_{k i_k}(\zeta_k)} \right|.$$

It follows from (5) and (6.12) that $\left| \omega_{k i_k}(z_k) / \omega_{k i_k}(x_{k i_k}) \right| \leqq \Phi^{i_k}(z_k, E_k, 0)$, z_k being arbitrary. By Lemma 1 there is an $l_0 > 0$ such that

$$\left| \omega_{kl}(z_k) / \omega_{kl}(x_{kl}) \right| \geqq (R - \varepsilon)^l \text{ for } z_k \in C_k, \quad k = 1, 2 ..., n, \; l \geqq l_0.$$

On the other hand, since $C_k \cap E_k = \emptyset$ and C_k is bounded, there is a constant $\delta < 1$ such that

$$\left| \omega_{kl}(z_k) / \omega_{kl}(x_{kl}) \right| \geqq \delta(R - \varepsilon)^l, \quad z_k \in C_k, \quad k = 1,2 \ldots, n, \quad l = 1,2, \ldots, l_0.$$

Therefore,

$$\left| P_\nu^\mu(z) \right| \leqq (1 + \varepsilon)^\mu M^\mu M_1 \max_{1 \leqq k \leqq n} \{\Phi^{\mu\nu}(z_k, E_k, 0)\} \times \sum_{i_1 + \ldots + i_n \leqq \mu\nu} \frac{1}{\delta^n (R - \varepsilon)^{i_1 + \cdots + i_n}}$$

whence

$$\left| P_\nu^\mu(z) \right| \leqq (1 + \varepsilon)^\mu M^\mu M_1 \delta^{-n C_{\mu\nu} + n^n} (R - \varepsilon)^{-\mu\nu} \max_{1 \leqq k \leqq n} \{\Phi^{\mu\nu}(z_k, E_k, 0)\}, \quad z \in C^n.$$

After taking the μth root of both sides of this inequality and letting μ go to ∞, we shall find

$$\left| P_\nu(z) \right| \leqq \frac{1 + \varepsilon}{(R - \varepsilon)^\nu} M \max_{1 \leqq k \leqq n} \{\Phi^\nu(z_k, E_k, 0)\}, \quad z \in C^n.$$

Since $\varepsilon > 0$ may be arbitrarily small and R may be arbitrarily close to 1, we conclude that inequality (4) holds.

THEOREM 1. *If* $E = E_1 \times \ldots \times E_n$, *then*

$$(8) \qquad\qquad \Phi(z, E, 0) = \max_{1 \leqq k \leqq n} \{\Phi(z_k, E_k, 0)\}, \quad z \in C^n.$$

Proof. If $P_\nu(z_k)$ is a polynomial in z_k of degree ν, then it is a polynomial of the same degree in $z = (z_1, \ldots, z_n)$, whence by (6.12)

$$\left| P_\nu(z_k) \right| \leqq (\max_{z \in E} \left| P_\nu(z_k) \right|) \phi^\nu(z, E, 0), \quad z \in C^n.$$

Therefore due to Theorem 2, §6,

$$(9) \qquad\qquad \Phi(z_k, E_k, 0) \leqq \Phi(z, E, 0), \quad k = 1, 2, \ldots, n.$$

On the other hand, by Lemma 2 and in virtue of Theorem 2, §6, we have

$$(10) \qquad\qquad \Phi(z, E, 0) \leqq \max_{1 \leqq k \leqq n} \{\Phi(z_k, E_k, 0)\}, \quad z \in C^n.$$

Now (8) follows immediately from (9) and (10).

COROLLARY 1. *If* $d(E_k) > 0$, $k = 1, 2, \ldots, n$, *then* $\log \Phi^*(z, E, 0)$, *where* $E = E_1 \times \ldots \times E_n$, *is plurisubharmonic in* C^n.

COROLLARY 2. *If* E_k, $k = 1, 2, \ldots, n$, *is regular and* $E = E_1 \times \ldots \times E_n$, *then* $\Phi(z, E, 0)$ *is continuous in* C^n.

REMARK. It follows from (8) that if E_{k_0} for some $1 \leqq k_0 \leqq n$ is not regular, then $\Phi(z, E, 0)$, $E = E_1 \times \ldots \times E_n$, is discontinuous at some points outside of E, e.g., if $\Phi(z_1, E_1, 0)$ is not continuous at a point $z_1^0 \in E_1$, then the function $\Phi(z, E, 0)$ is discontinuous at any point $(z_1^0, z_2, \ldots, z_n)$, where $(z_2, \ldots, z_n) \notin E_2 \times \ldots \times E_n$.

9. **The function** $\Phi(z, E, 0)$ **for circular sets.** The set $E \subset C^n$ is called circular if along with the point $z^0 = (z_1^0, \ldots, z_n^0) \in E$ all the points of the circle

(1) $z = e^{i\theta} z^0 \equiv (e^{i\theta} z_1^0, \ldots, e^{i\theta} z_n^0),$ $0 \leqq \theta \leqq 2\pi,$

belong to E.

The set $E \subset C^n$ is called a Reinhardt circular set if along with the point $z^0 = (z_1^0, \ldots, z_n^0) \in E$ also the set

$$\{z \mid |z_k| = |z_k^0|, \qquad k = 1, 2, \ldots, n\}$$

belongs to E.

Let E be a bounded closed subset of C^n, unisolvent with respect to homogeneous polynomials. The function $b(z)$ being defined and lower semicontinuous on E, let

(2) $h^{(\nu)} = \{h_1^{(\nu)}, \ldots, h_\nu^{(\nu)}\},$ $\nu_0 = C_{\nu+n-1, n-1}$

be the νth extremal system of E defined by (5.5) and (5.6). If $p^{(\nu)} = \{p_1, \ldots, p_{\nu_0}\}$ is an arbitrary unisolvent system of points of E, then the functions

(3) $\psi^{(i)}(z, p^{(\nu)}, b) = T^{(i)}(z, p^{(\nu)}) e^{\nu b(p_i)},$ $i = 1, 2, \ldots, \nu_0,$

$T^{(i)}(z, p^{(\nu)})$ denoting the polynomial (2.12), are homogeneous polynomials of degree ν. Define the extremal functions $\psi_\nu^{(i)}(z, E, b)$, $i = 1, 2, 3, 4$, corresponding to E and b by the formulas

(4) $\psi_\nu^{(1)}(z, E, b) = \max_{(i)} |\psi^{(i)}(z, h^{(\nu)}, b)|,$ $z \in C^n,$

(5) $\psi_\nu^{(2)}(z, E, b) = \sum_{i=1}^{\nu_0} |\psi^{(i)}(z, h^{(\nu)}, b)|,$ $z \in C^n,$

(6) $\psi_\nu^{(3)}(z, E, b) = \inf_{p^{(\nu)} \subset E} \left\{ \max_{(i)} |\psi^{(i)}(z, p^{(\nu)}, b)| \right\},$ $z \in C^n,$

(7) $\psi_\nu^{(4)}(z, E, b) = \inf_{p^{(\nu)} \subset E} \left\{ \sum_{i=1}^{\nu_0} |\psi^{(i)}(z, p^{(\nu)}, b)| \right\},$ $z \in C^n.$

By reasoning quite analogous to the reasoning of §6 we may prove (see also [19])

THEOREM 1. *At any point $z \in C^n$ the sequences $\{(\psi_\nu^{(i)}(z, E, b)^{1/\nu}\}$, $i = 1, 2, 3, 4,$ are convergent to the same limit $\psi(z, E, b)$,*

(8) $\psi(z, E, b) = \lim_{\nu \to \infty} (\psi^{(i)}(z, E, b))^{1/\nu},$ $z \in C^n,$ $i = 1, 2, 3, 4.$

Lemma 1 of §6 now takes the form of

LEMMA 1. *If $Q_\nu(z)$ denotes an arbitrary homogeneous polynomial of degree ν such that*

$$|Q_\nu(z)| \leqq M \exp[\nu b(z)], \qquad z \in E, \quad M = \text{const},$$

then

$$|Q_\nu(z)| \leqq M \psi^\nu(z, E, b), \qquad z \in C^n.$$

One obtains easily also the following properties of ψ.

1° Function $\psi(z, E, b)$ is given by

$$\psi(z, E, b) = \lim_{\nu \to \infty} \{ \sup_{Q_\nu \in A_\nu} |Q_\nu(z)|^{1/\nu} \}, \qquad z \in C^n,$$

where $A_v = A_v(E, b)$ denotes the family of all homogeneous polynomials $Q_v(z)$ of degree v such that $|Q_v(z)| \leqq \exp[vb(z)]$ for $z \in E$.

2° $\psi(z, E, b)$ is an absolutely homogeneous function of order 1, i.e., $\psi(\lambda z, E, b) = |\lambda| \psi(z, E, b)$ for $z \in C^n$ and for any complex λ.

3° There exists a positive number m which depends only on E and b such that

$$\psi(z, E, b) \geqq m(|z_1| + \dots + |z_n|), \qquad z \in C^n.$$

4° If $\psi(z, E, b)$ is bounded on the unit sphere $\{z \mid \|z\| \leqq 1\}$ and $\psi^*(z, E, b) = \lim\sup_{z' \to z} \psi(z', E, b)$, then $\log \psi^*(z, E, b)$ is plurisubharmonic in C^n.

5° If E_k, $k = 1, 2, \dots, n$, is a bounded closed set of positive logarithmic capacity, $z_1^0 \neq 0$ is a fixed point of the (z_1)-plane and $E = \{z_1^0\} \times E_2 \times \dots \times E_n$, then there is a number $M > 0$ such that

$$\psi(z, E, b) \leqq M(|z_1| + \dots + |z_n|), \qquad z \in C^n.$$

Given an arbitrary compact set $E \subset C^n$, let $H(z) = H(z, E)$ be defined by

$$(9) \qquad H(z) = \lim_{v \to \infty} \{\sup(|a_{\mu_1 \dots \mu_n} z_1^{\mu_1} \dots z_k^{\mu}|)^{1/v}\}, \qquad z \in C^n,$$

where sup is taken over all the monomials $a_{\mu_1 \dots \mu_n} z_1^{\mu_1} \dots z_n^{\mu_n}$ of degree $v = \mu_1 + \dots + \mu_n$ such that

$$(10) \qquad |a_{\mu_1 \dots \mu_n} z_1^{\mu_1} \dots z_n^{\mu_n}| \leqq 1 \quad \text{for} \quad z \in E.$$

THEOREM 2. *If $E \subset C^n$ is a compact Reinhardt circular set, then*

$$(11) \qquad \Phi(z, E, 0) = \max(1, \psi(z, E, 0)) = \max(1, H(z)), \qquad z \in C^n.$$

Proof. Let

$$P_v(z) = \sum_{k=0}^{v} \left(\sum_{\mu_1 + \dots + \mu_n = k} a_{\mu_1 \dots \mu_n} z_1^{\mu_1} \dots z_n^{\mu_n} \right)$$

be a polynomial of degree v such that

$$|P_v(z)| \leqq 1 \quad \text{for} \quad z \in E.$$

Then by the Cauchy inequalities we have (due to the definition of a Reinhardt circular set)

$$|a_{\mu_1 \dots \mu_n} z_1^{\mu_1} \dots z_n^{\mu_n}| \leqq 1 \quad \text{for} \quad z \in E.$$

Therefore, for the points z such that $H(z) \neq 1$ we have

$$|P_v(z)| \leqq \sum_{k=0}^{v} C_{n+k-1, k} H^k(z) \leqq (v + n)^n \sum_{k=0}^{v} H^k(z) = (v + n)^n \frac{H^{v+1}(z) - 1}{H(z) - 1}.$$

Of course, we have also

$$|P_v^\mu(z)| \leqq (\mu v + n)^n \frac{H^{\mu v+1}(z) - 1}{H(z) - 1}, \qquad \mu = 1, 2, \dots, \quad H(z) \neq 1,$$

whence

$$(12) \quad |P_v(z)| \leqq \lim_{\mu \to \infty} \left[(\mu v + n)^n \frac{H^{\mu v+1}(z) - 1}{H(z) - 1} \right]^{1/\mu} = \begin{cases} H^v(z), & \text{if } H(z) > 1, \\ 1, & \text{if } H(z) < 1. \end{cases}$$

Thus, due to Theorem 2, §6, we have

$$\Phi(z,E,0) \leqq \max(1, H(z)), \qquad z \in C^n.$$

On the other hand, it follows from Theorem 2, §6, from 1° of this section and from the definition of $H(z)$ that

(13)
$$H(z) \leqq \psi(z,E,0) \leqq \Phi(z,E,0), \qquad z \in C^n.$$

Now (11) follows directly from (12) and (13).

THEOREM 3. *If $E \subset C^n$ is a compact circular set, then*

(14)
$$\Phi(z,E,0) = \max(1, \psi(z,E,0)), \qquad z \in C^n.$$

Proof. If

$$P_\nu(z) = \sum_{k=0}^{\nu} \sum_{\mu_1 + \ldots + \mu_n = k} a_{\mu_1 \cdots \mu_n} z_1^{\mu_1} \ldots z_n^{\mu_n} = \sum_{k=0}^{\nu} Q_k(z)$$

is an arbitrary polynomial of degree ν such that

$$|P_\nu(z)| \leqq 1 \quad \text{for} \quad z \in E,$$

then the function $\omega_\nu(\lambda) = \sum_{k=0}^{\nu} Q_k(\lambda \dot z) = \sum_{k=0}^{\nu} Q_k(\dot z)\lambda^k$ is a polynomial in λ of degree ν. If $\dot z \in E$, then $|\omega_\nu(\lambda)| \leqq 1$ for $|\lambda| = 1$. Therefore, by the Cauchy inequalities

$$|Q_k(\dot z)| \leqq 1, \qquad \dot z \in E, \quad k = 0,1,\ldots,\nu,$$

whence due to Lemma 1

$$|Q_k(z)| \leqq \psi^k(z,E,0), \qquad z \in C^n, \quad k = 0,1,\ldots,\nu.$$

Therefore,

$$|P_\nu(z)| \leqq \frac{\psi^{\nu+1}(z)-1}{\psi(z)-1}, \quad \text{as} \quad \psi(z,E,0) \neq 1, \quad \nu = 1,2,\ldots.$$

By a familiar reasoning this inequality, along with the fact that any homogeneous polynomial of degree ν is also an ordinary polynomial of degree ν, implies (14)

REMARK. One may prove [19] that $H(z)$ is an upper envelope of all absolutely homogenous functions of order 1 which are $\leqq 1$ on E and which are convex with respect to $\xi_k = \log|z_k|$, $k = 1,2,\ldots,n$. Moreover, if E is a Reinhardt circular set such that for any $z = (z_1,\ldots,z_n) \in E$ we have $z_i \neq 0$, $i = 1,\ldots,n$, then $H(z) = H(z,E)$ is continuous in C^n.

EXAMPLES. 1. If $E = \{z \mid \|z\| = r\}$, then $H(z) = \psi(z) = \|z\|/r$, $\Phi(z) = \max(1, \|z\|/r)$, $z \in C^n$.

2. If $E = \{z \mid \||z_1/a_1|^\alpha + \ldots + |z_n/a_n|^\alpha = r\}$, then

$$H(z) = \psi(z) = \left[\frac{1}{r} \sum_{i=1}^{n} \left|\frac{z_i}{a_i}\right|^\alpha\right]^{1/\alpha}, \qquad z \in C^n.$$

3. If $E = \{(z_1,z_2) \mid |z_1| = \xi_1, |z_2| = \eta_1\} \cup \{(z_1,z_2) \mid |z_1| = \xi_2, |z_2| = \eta_2\}$, $0 < \xi_1 < \xi_2$, $0 < \eta_2 < \eta_1$, then there exist constants α, β and γ such that

$$H(z) = \psi(z) = \max \left(\frac{|z_1|}{\xi_2}, \frac{|z_2|}{\eta_1}, \frac{|z_1|^\alpha |z_2|^\alpha}{\gamma} \right), \quad z \in C^2.$$

10. Generalization of the Bernstein-Walsh theorems. In this section we shall always assume that E is a compact subset of C^n such that the function $\Phi(z) = \Phi(z, E, 0)$ is continuous in C^n and $CE = D_\infty$.

Given any $\rho > 1$ we define E_ρ and and C_ρ by

(1) $$E_\rho = \{z \mid \Phi(z) < \rho\}, \quad C_\rho = \{z \mid \Phi(z) = \rho\}.$$

Since by assumption $\Phi(z)$ is continuous, then E_ρ is open and C_ρ is the boundary of E_ρ, because $\Phi(z)$ being plurisubharmonic in C^n cannot attain its maximum in the interior of a domain without being constant. But $\Phi(z) = \Phi(z, E, 0) > \|z\|/r$ for sufficiently large $r > 0$, so $\Phi(z) \not\equiv$ const.

Given the function $f(z)$ defined and bounded on E, denote by R the largest real number such that there exists a function $F(z)$ holomorphic in E_R and equal to $f(z)$ on E.

We say that the sequence of polynomials $\{P_\nu(z)\}$, where P_ν is of degree ν, converges maximally to $f(z)$ on E, if

(2) $$\limsup_{\nu \to \infty} (\max_{z \in E} |f(z) - P_\nu(z)|)^{1/\nu} = \frac{1}{R}.$$

THEOREM 1. *If the polynomials $P_\nu(z)$ of respective degrees ν satisfy the condition*

(3) $$\limsup_{\nu \to \infty} |f(z) - P_\nu(z)|^{1/\nu} \leqq \frac{1}{R}, \qquad R > 1, \quad z \in E,$$

and if $R_1 \in (1, R)$, then the sequence $\{P_\nu(z)\}$ is uniformly convergent in E_{R_1}.

Proof. The difference $P_{\nu+1} - P_\nu$ is a polynomial of degree $\nu + 1$; therefore by Lemma 1, §6,

(4) $$|P_{\nu+1}(z) - P_\nu(z)| \leqq [\max_{z \in E} |P_{\nu+1}(z) - P_\nu(z)|]\Phi^{\nu+1}(z), \quad z \in C^n.$$

But

(5) $$\max_{z \in E} |P_{\nu+1}(z) - P_\nu(z)| \leqq \max_{z \in E} |P_{\nu+1} - f| + \max_{z \in E} |P_\nu - f|.$$

If $\varepsilon > 0$ is so small that $(R_1/R)(1 + \varepsilon) < 1$, then for sufficiently large N, we have

$$\max_{z \in E} |f - P_\nu| \leqq \left(\frac{1 + \varepsilon}{R} \right)^\nu, \quad \nu > N,$$

and further, by (4) and (5),

$$|P_{\nu+1} - P_\nu| \leqq 2 \left(\frac{1 + \varepsilon}{R} \right)^\nu \Phi^{\nu+1}(z), \quad z \in C^n, \quad \nu \geqq N,$$

whence

$$|P_{\nu+1} - P_\nu| \leqq 2R_1 \left[\frac{(1 + \varepsilon)R_1}{R} \right]^\nu, \quad z \in C_{R_1}, \qquad \nu \geqq N.$$

Therefore, the series $P_0 + \sum_{k=0}^\infty (P_{k+1} - P_k)$ converges uniformly in $\bar{E}_{R_1}$. Since $P_0 + \sum_{k=0}^\nu (P_{k+1} - P_k) = P_\nu$, the proof is completed.

For polynomials in one variable Theorem 1 is due to Bernstein and Walsh (see [27, p. 78]).

REMARK. After having Lemma 1, §6, the proof of Theorem 1 is the same as the proof of the corresponding theorem for polynomials in one variable. However, the proof of the lemma differs from the well-known proof of the corresponding lemma in the theory of one variable, which is based on the maximum principle for harmonic functions.

THEOREM 2. *If $f(z)$ is a holomorphic function on E, then there exists a sequence of polynomials $P_\nu(z)$ of respective degrees ν which converges maximally to $f(z)$.*

Proof. 1° Suppose E is circular. By Theorem 9.3 and due to 2°, §9, E_R is also a circular domain. If the function $f(z)$ is holomorphic in E_R, then it may be developed in a series of homogeneous polynomials

$$(6) \qquad f(z) = \sum_{\nu=0}^{\infty} Q_\nu(z), \qquad z \in E_R,$$

$Q_\nu(z)$ being a homogeneous polynomial of degree ν. If $R_1 \in (1,R)$, then $\bar{E}_{R_1} = E_{R_1} + C_{R_1}$ is a compact subset of E_R and the series (6) is uniformly absolutely convergent on $\bar{E}_{R_1}$. Therefore, there is a constant $M > 0$ such that

$$\left| Q_\nu(z) \right| \leq M, \qquad \nu = 0, 1, \dots, \quad z \in E_{R_1}.$$

In virtue of Lemma 1, §9, we have

$$\left| Q_\nu(z) \right| \leq M \psi^\nu(z, \bar{E}_{R_1}, 0), \qquad z \in C^n, \quad \nu = 0, 1, \dots,$$

whence due to the homogeneity of $\psi(z, \bar{E}_{R_1}, 0)$ and $\psi(z, E, 0)$ we have

$$\left| Q_\nu(z) \right| \leq M \frac{1}{R_1^\nu} \psi^\nu(z, E, 0), \qquad z \in C^n.$$

Let $P_\nu(z) = \sum_{k=0}^{\nu} Q_k(z)$. Then

$$(7) \quad \left| f(z) - P_\nu(z) \right| = \left| \sum_{k=\nu+1}^{\infty} Q_k(z) \right| \leq M \sum_{k=\nu+1}^{\infty} \frac{1}{R_1^k} = \frac{M}{R_1^{\nu+1}(1 - 1/R_1)}, \quad z \in E,$$

whence due to the arbitrariness of $R_1 \in (1,R)$, we have

$$\lim_{\nu \to \infty} \sup (\max_{z \in E} |f - P_\nu|)^{1/\nu} \leq \frac{1}{R},$$

and the inequality $\lim \sup_{\nu \to \infty} (\max_{z \in E} |f - P_\nu|)^{1/\nu} < 1/R$ is impossible, as follows immediately from Theorem 1, Q.E.D.

2° Suppose E is arbitrary. Since by our general assumption the function $\Phi(z) = \Phi(z, E, 0)$ is continuous and, on the other hand, $\Phi(z)$ is an upper bound of continuous functions $\Phi_\nu(z) = (\max_{(i)} |L^{(i)}(z, \gamma^{(\nu)})|)^{1/\nu}$, therefore by Dini's theorem the sequence Φ_ν converges uniformly to $\Phi(z)$ on any closed bounded subset of C^n.

Let $R_1 > 1$ and $\varepsilon > 0$ be arbitrary real numbers such that $R_1 + \varepsilon < R - \varepsilon$. Since $\Phi_\nu(z)$ converges to $\Phi(z)$ uniformly on $\bar{E}_R$, there exists integer m such that

$$(8) \qquad \Phi_m(z) < R_1 + \varepsilon \qquad \text{for } z \in C_{R_1}$$

and

$$(9) \qquad \Phi_m(z) > R - \varepsilon \qquad \text{for } z \in C_R.$$

Let

$$G = \{z\,\big|\,|L^{(i)}(z,\gamma^{(m)})| < (R - \varepsilon)^m, \quad i = 1,2,\ldots,m_*\} \ , \quad m_* = C_{m+n,n}$$

Of course,

$$G = \{z\,|\,\Phi_m(z) < R - \varepsilon\}.$$

In virtue of (8) and (9) and since $R_1 + \varepsilon < R - \varepsilon$, we have

$$(10) \qquad C_{R_1} \subset G \quad \text{and} \quad \bar{G} \subset E_R.$$

Let m' denote the smallest integer such that G' defined by

$$G' = \{z\,\big|\,|L^{(i_k)}(z,\gamma^{(m)})| < (R - \varepsilon)^m, \quad k = 1,2,\ldots,m'\}$$

is identical with G. Without any loss of generality we may assume that $i_k = k$ for $k = 1,2,\ldots,m'$. Thus

$$(11) \qquad G = \{z\,\big|\,|L^{(i)}(z,\gamma^{(m)})| < (R - \varepsilon)^m, \quad i = 1,2,\ldots,m'\}.$$

By assumption (by definition of R) the function $f(z)$ is holomorphic in E_R. Then it is holomorphic by (10) in $\bar{G}$. In the following we shall write $L^{(i)}(z)$ instead of $L^{(i)}(z,\gamma^{(m)})$. By a theorem of A. Weil [25; 26],

$$f(z) = \frac{1}{(2\pi i)^n} \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq m'} \int_{\sigma_{i_1 \ldots i_n}} \frac{\delta_{i_1 \ldots i_n}(z,\zeta) f(\zeta) d\zeta_1 \ldots d\zeta_n}{\prod_{k=1}^{m'}[L^{(i_k)}(z) - L^{(i_k)}(\zeta)]}, \ z \in G,$$

or

$$(12) \quad f(z) = \sum_{1 \leq i_1 < \ldots < i_n \leq m'} \sum_{k_1,\ldots,k_n = 0}^{\infty} P_{i_1 \ldots i_n k_1 \ldots k_n}(z)[L^{(i_1)}(z)]^{k_1} \ldots [L^{(i_n)}(z)]^{k_n}, \quad z \in G,$$

where

$$(13) \qquad P_{i_1 \ldots i_n k_1 \ldots k_n}(z) = \frac{1}{(2\pi i)^n} \int_{\sigma_{i_1 \ldots i_n}} \frac{\delta_{i_n \ldots i_n}(z,\zeta) f(\zeta) d\zeta_1 \ldots d\zeta_n}{[L^{(i_n)}(\zeta)]^{k_n+1} \ldots [L^{(i_n)}(\zeta)]^{k_n+1}}$$

and $\sigma_{i_1 \ldots i_n}$ denotes some n-dimensional part of the boundary of G and $\delta_{i_1 \ldots i_n}(z,\zeta)$ is the determinant

$$\delta_{i_1 \ldots i_n} = \det[P_{i_k l}]_{k,l = 1,2,\ldots n} \ ,$$

while $P_{i_k l}$ is defined by the relations

$$L^{(i)}(z) - L^{(i)}(\zeta) \equiv \sum_{l=1}^{n} (z_l - \zeta_l) P_{il} \ , \qquad i = 1,2,\ldots,m'.$$

Thus, $P_{il}(z,\zeta)$ are polynomials of order m, both in $z = (z_1,\ldots,z_n)$ and in $\zeta = (\zeta_1,\ldots\zeta_n)$. The polynomial

$$Q_{k_1 \ldots k_n}(z) = \sum_{1 \leq i_1 \leq \ldots \leq i_n \leq m'} P_{i_1 \ldots i_n k_1 \ldots k}(z)[L^{(i_1)}(z)]^{k_1} \ldots [L^{(i_n)}(z)]^{k_n}$$

is of order $m + m(k_1 + \ldots + k_n)$. For $z \in E$ we have $|L^{(i)}(z)| = |L^{(i)}(z, \gamma^{(m)})| \leqq 1$, $i = 1, 2 \ldots, m_*$. Therefore, due to (10), (11) and (13), we have

(14) $$|Q_{k_1 \ldots k_n}(z)| \leqq M/(R - \varepsilon)^{m(k_1 + \ldots + k_n + n)}, \qquad z \in E,$$

M being a constant which depends only on R_1 and f. Let

$$P_{m(\nu+n)}(z) = \sum_{k_1 + \ldots + k_n \leqq \nu} Q_{k_1 \ldots k_n}(z), \qquad \nu = 1, 2, \ldots.$$

By (12) and (14) we have

$$|f(z) - P_{m(\nu+n)}(z)| \leqq M \sum_{k_1 + \ldots + k_n \geqq \nu + 1} 1/(R - \varepsilon)^{m(k_1 + \ldots + k_n + n)}$$

$$\leqq M_1/(R - \varepsilon)^{m(\nu + 1 + n)}, \qquad z \in E,$$

M_1 being a constant. Therefore,

$$\limsup_{\nu \to \infty} \{ \max_{z \in E} |f - P_{m(\nu+n)}| \}^{1/m(\nu+n)} \leqq \frac{1}{R - \varepsilon}.$$

The polynomials $\tilde{P}_\mu$, where $\tilde{P}_\mu = P_{m(\nu+n)}$ for $m(\nu + n) \leqq \mu < m(\nu + n), \nu = 1, 2, \ldots,$ and $\tilde{P}_\mu \equiv 0$ for $\mu = 1, 2, \ldots, mn$, are of respective degrees μ and moreover

$$\limsup_{\nu \to \infty} (\max_{z \in E} |f - \tilde{P}_\nu|)^{1/\nu} \leqq \frac{1}{R - \varepsilon}.$$

The sequence $\{\tilde{P}_\nu\}$ depends on ε and R_1. Letting now R_1 go to R and ε to 0 we may find by a diagonal process polynomials $P_\nu(z)$ of respective degrees ν such that

$$\limsup_{\nu \to \infty} (\max_{z \in E} |f - P_\nu|)^{1/\nu} \leqq \frac{1}{R}.$$

Since by Theorem 1 the inequality $\limsup_{\nu \to \infty} (\ldots)^{1/\nu} < 1/R$ cannot hold, the proof of the theorem is completed.

In the case of one variable Theorem 2 is due to Faber, Bernstein, Szegö and Walsh (for reference see [27]).

Let $\pi_\nu(z)$ denote the Tchebycheff polynomial of degree ν of the best approximation to $f(z)$ on E. An immediate consequence of Theorems 1 and 2 is

THEOREM 3. *A necessary and sufficient condition that the function $f(z)$ be holomorphic in E_ρ, $\rho > 1$, is that*

$$\limsup_{\nu \to \infty} (\max_{z \in E} |f(z) - \pi_\nu(z)|)^{1/\nu} \leqq \frac{1}{\rho}.$$

For polynomials in one variable this theorem is well known [27]. In the case that E is a Cartesian product of linear intervals $E_k = \{z_k | -1 \leqq z_k = x_k \leqq 1\}$, $k = 1, 2, \ldots, n$, Theorem 3 is due to Sapogov [17].

11. Interpolation at extremal points.

THEOREM 1. *If the complement of E is connected, $\Phi(z) = \Phi(z, E, 0)$ is continuous in C^n and the function $f(z)$ is holomorphic on E, then the sequence of interpolating polynomials*

(1) $$L_\nu(z,f) = \sum_{i=1}^{\nu_*} f(\gamma_i^{(\nu)}) L^{(i)}(z,\gamma^{(\nu)}), \qquad \nu = 1,2\ldots,$$

where $\gamma^{(\nu)} = \{\gamma_1^{(\nu)}, \ldots, \gamma_{\nu_*}^{\nu}\}$ is the νth extremal system of E with respect to $b(z) \equiv 0$ converges maximally to $f(z)$.

Proof. By Theorem 10.2 there is a sequence of polynomials $P_\nu(z)$ of respective degrees ν which converges maximally to $f(z)$, i.e.,

(2) $$\limsup_{\nu \to \infty} (\max_{z \in E} |f - P_\nu|)^{1/\nu} = \frac{1}{R},$$

where $R > 1$ is the largest number such that $f(z)$ is holomorphic in E_R. If $R_1 \subset (1,R)$, there is a const $M > 0$ such that

(3) $$|f(z) - P_\nu(z)| \leqq \frac{M}{R_1^\nu}, \qquad \nu = 1,2,\ldots, \quad z \in E.$$

Since

$$L_\nu(z,f) - P_\nu(z) \equiv \sum_{i=1}^{\nu_*} [f(\gamma_i^{(\nu)}) - P_\nu(\gamma_i^{(\nu)})] L^{(i)}(z,\gamma^{(\nu)}), \qquad \nu = 1,2,\ldots,$$

then by (3) and (6.7)

(4) $$|L_\nu(z,f) - P_\nu(z)| \leqq \frac{\alpha_\nu \nu_* M}{R_1^\nu}, \qquad \nu = 1,2,\ldots, \qquad z \in E.$$

It follows from (3) and (4) that

$$|f(z) - L_\nu(z,f)| \leqq \frac{M}{R_1^\nu}(1 + \nu_* \alpha_\nu), \qquad \nu = 1,2,\ldots, \quad z \in E,$$

whence by (5.2) we have

$$\limsup_{\nu \to \infty} (\max_{z \in E} |f(z) - L_\nu(z,f)|)^{1/\nu} \leqq \frac{1}{R_1}.$$

Due to the arbitrariness of R_1 and because of Theorem 10.1 the last inequality implies our theorem, Q.E.D.

If $E \subset C^1$, Theorem 1 is due to Fekete [5] (see also [27, p. 171]).

Let now $E = E_1 \times \ldots \times E_n$, where E_k is regular and has a connected complement. Let

(5) $$\omega_{k\nu}(z_k) = (z_k - x_{k0})\ldots(z_k - x_{k\nu-1}), \qquad \nu = 1,2,\ldots, \quad k = 1,2,\ldots,n,$$

where

(6) $$x_{k0}, x_{k1}, \ldots, x_{k\nu}, \ldots,$$

denotes a sequence of Leja's extremal points of E_k, $k = 1,2,\ldots,n$, respectively (see Lemma 8.1). Suppose $f(z)$ is holomorphic in the Cartesian product $E_{R_1 \ldots R_n}$ of the domains

(7) $$E_{R_k} = \{z_k \mid \Phi(z_k, E_k, 0) < R_k\}, \qquad R_k > 1, \quad k = 1,2,\ldots,n.$$

Let

(8) $$C_k = \{z_k \mid \Phi(z_k, E_k, 0) = R_k'\}, \qquad \text{where } R_k' \in (1, R_k), \quad k = 1,2,\ldots,n.$$

Theorem 2. *The Newton's series of* $f(z)$

$$(9) \qquad f(z) \sim a_0 + \sum_{l=1}^{\infty} \sum_{i_1+\ldots+i_n=l} a_{i_1\ldots i_n} \prod_{k=1}^{n} \omega_{ki_k}(z_k) ,$$

converges uniformly on any closed subset of $E_{R_1\ldots R_n}$.

Proof. Since by (3.10) we have

$$a_{i_1\ldots i_n} = \frac{1}{(2\pi i)^n} \int_{C_1}\cdots\int_{C_n} \frac{f(\zeta_1,\ldots,\zeta_n)d\zeta_1\ldots d\zeta_n}{\prod\limits_{k=1}^{n}(\zeta_k - x_{ki_k}) \prod\limits_{k=1}^{n}\omega_{ki_k}(\zeta_k)} ,$$

and $C_k \cap E = \emptyset$, $k = 1,2,\ldots,n$, then

$$\left|a_{i_1\ldots i_n}\right| \leqq M \Big/ \prod_{k=1}^{n} \min_{\zeta_k \in C_k} \left|\omega_{ki_k}(\zeta_k)\right|, \qquad i_1,i_2,\ldots,i_n = 0,1,\ldots,$$

M depending only on $C_1 \times C_2 \times \ldots \times C_n$ and on f. We know (see Lemma 8.1) that the sequence $\{(\left|\omega_{k\nu}(z_k)/\omega_{k\nu}(x_{k\nu})\right|)^{1/\nu}\}$ converges uniformly to $\Phi(z_k,E_k,0)$ on any compact subset of the (z_k)-plane. Therefore, given $\varepsilon_k > 0$ sufficiently small, there is a constant $M_1 > 0$ such that

$$\left|a_{i_1\ldots i_n}\right| \leqq M_1 \Big/ \left[\prod_{k=1}^{n}(R_k' - \varepsilon_k)^{i_k}\left|\omega_{ki_k}(x_{ki_k})\right|\right],$$

whence it follows that series (9) is majorized by the series

$$(10) \qquad \left|a_0\right| + M_1 \sum_{l=1}^{\infty} \sum_{i_1+\ldots+i_n=l} \prod_{k=1}^{n} \frac{\left|\omega_{ki_k}(z_k)\right|}{\left|\omega_{ki_k}(x_{ki_k})\right| R_k' - \varepsilon_k)^{i_k}}.$$

Let $\varepsilon_k = \varepsilon_1(R_1'/R_k')$, $k = 2,\ldots,n$, then

$$\frac{R_k' - 2\varepsilon_k}{R_k' - \varepsilon_k} = \frac{R_1' - 2\varepsilon_1}{R_1' - \varepsilon_1}.$$

There is a constant $M_2 > 0$ such that

$$\left|\frac{\omega_{ki_k}(z_k)}{\omega_{ki_k}(x_{ki_k})}\right| \leqq M_2(R_k' - 2\varepsilon_k)^{i_k} \text{ for } z_k \in \{z_k \mid \Phi(z_k,E_k,0) = R_k' - 3\varepsilon_k\}.$$

Then the series

$$\left|a_0\right| + M_1 M_2 \sum_{l=1}^{\infty} \sum_{i_1+\ldots+i_n=l} \left(\frac{R_1' - 2\varepsilon_1}{R_1' - \varepsilon_1}\right)^{i_1+\ldots+i_n}$$

is convergent and it majorizes the series (10) (and therefore the series (9) for z in the Cartesian product of the sets $\{z_k \mid \Phi(z_k,E_k,0) = R_k' - 3\varepsilon_k\}$. Due to the arbitrariness of ε_k and of R_k', $k = 1,\ldots,n$, this implies that the series (9) is convergent uniformly on any closed subset of $E_{R_1\ldots R_n}$ to some holomorphic function $g(z)$. But due to Lemma 3.2, we have

$$g(x_{1i_1},\ldots,x_{ni_n}) = f(x_{1i_1},\ldots,x_{ni_n}), \qquad i_1,i_2,\ldots,i_n = 0,1,2,\ldots,$$

whence $g(z) \equiv f(z)$. The proof is completed.

12. **Existence of the limit** $u(z,E,b) = \lim_{\lambda \downarrow 0} (1/\lambda) \log \Phi(z,E,\lambda b)/\Phi(z,E,0)$. Let E be a compact set in C^n such that $\Phi(z,E,0)$ is finite at any point of C^n. If $b(z)$ is an arbitrary real function defined and bounded on E and if λ is a real number > 0, then by (7.1)

$$\inf_{\zeta \in E} b(\zeta) = b_0 \leqq \frac{1}{\lambda} \log \frac{\Phi(z,E,\lambda b)}{\Phi(z,E,0)} \leqq B_0 = \sup_{\zeta \in E} b(\zeta), \quad z \in C^n.$$

We shall prove the following inequality

(1) $$\left[\frac{\Phi(z,E,\lambda b)}{\Phi(z,E,0)} \right]^{1/\lambda} \leqq \left[\frac{\Phi(z,E,\lambda' b)}{\Phi(z,E,0)} \right]^{1/\lambda'}, \quad 0 < \lambda' \leqq \lambda, z \in C^n.$$

At first let λ and λ' be rational

$$\lambda' = \frac{p'}{q'} < \frac{p}{q} = \lambda.$$

We have

$$\frac{p'}{q'} b = \frac{p'q}{q'p} \cdot \frac{p}{q} b = \frac{1}{q'p} \left[p'q \frac{p}{q} b + (q'p - p'q)0 \right].$$

Therefore, by (7.8)

$$\Phi^{p'q}(z,E,\frac{p}{q}b)\Phi^{q'p-p'q}(z,E,0) \leqq \Phi^{q'p}(z,E, \frac{p'}{q'}b),$$

whence we obtain (1) for $\lambda = p/q$ and $\lambda' = p'/q'$.

To prove (1) for arbitrary λ and λ' let λ_ν and λ'_ν be rational numbers such that

$$\lambda'_\nu \leqq \lambda' \leqq \lambda \leqq \lambda_\nu , \quad \lambda'_\nu \to \lambda' , \quad \lambda_\nu \to \lambda,$$

and let $b_0 = \inf_{z \in E} b(z)$. We have $b(z) + b_0 \geqq 0$. Due to (7.7)

$$\left[\frac{\Phi(z,E,\lambda(b + b_0))}{\Phi(z,E,0)} \right]^{1/\lambda_\nu} \leqq \left[\frac{\Phi(z,E,\lambda_\nu(b + b_0))}{\Phi(z,E,0)} \right]^{1/\lambda_\nu}$$

and

$$\left[\frac{\Phi(z,E,\lambda'_\nu(b + b_0))}{\Phi(z,E,0)} \right]^{1/\lambda'} \leqq \left[\frac{\Phi(z,E,\lambda'(b + b_0))}{\Phi(z,E,0)} \right]^{1/\lambda'_\nu} .$$

Since λ_ν and λ'_ν are rational, we have by (7.5)

$$\exp\left(\frac{\lambda}{\lambda_\nu}b_0\right) \left[\frac{\Phi(z,E,\lambda b)}{\Phi(z,E,0)} \right]^{1/\lambda_\nu} \leqq \exp\left(\frac{\lambda'}{\lambda'_\nu}b_0\right) \left[\frac{\Phi(z,E,\lambda'b)}{\Phi(z,E,0)} \right]^{1/\lambda'_\nu},$$

whence the inequality (1) follows in an obvious way.

THEOREM 1. *If $E \subset C^n$ is a compact set, and if $b(z)$ is a real function defined and bounded on E, then there exists a finite limit*

(2) $$u(z,E,b) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \log \frac{\Phi(z,E,\lambda b)}{\Phi(z,E,0)}$$

at any point $z \in C^n$ such that $\Phi(z,E,0)$ is finite. Moreover, the function

$$u^*(z) = \limsup_{z' \to z} u(z',E,b)$$

is plurisubharmonic at any interior point of

$$\mathscr{E}_1 = \{z | \Phi(z,E,0) = 1\}.$$

Proof. The existence of the limit (2) follows directly from (1). The function $u^*(z)$ is, for $z \in \mathscr{E}_1$, an upper envelope of plurisubharmonic functions

$$\frac{1}{\lambda}\log \Phi^*(z,E,\lambda b) = \limsup_{z' \to z} \frac{1}{\lambda}\log \Phi(z',E,\lambda b);$$

therefore it is plurisubharmonic at any interior point if $\mathscr{E}_1$, Q.E.D.

If E is a line segment in C^1 and if E is a Jordan curve in C^1, the existence of the limit (2) for $z \in E$ has been proved in [11] and [8], respectively. The method of proof used by the authors of these papers was based on the generalized approximation theorem of Weierstrass.

Inoue in [8] and Leja in [13] have proved that if E is a Jordan curve in C^1 and $b(z)$ is continuous, then $u(z,E,b)$ is a solution of the Dirichlet boundary value problem for the interior of E with boundary values $b(z)$. The author of this paper has shown in his thesis [21] that if E is a boundary of a domain D which contains the point ∞ in its interior and if $b(z)$ is real and bounded, then $u(z,E,b)$ is Perron's generalized solution of the Dirichlet problem for any component of CE with boundary values $b(z)$.

In the next section we shall establish the relationship of $u(z,E,b)$ to Bremerman's [4] solution of the Dirichlet problem for plurisubharmonic functions in C^n.

13. The generalized Dirichlet problem for plurisubharmonic functions. Let D be a bounded domain in C^n and let $F = F(D)$ denote the Šilov boundary of D with respect to plurisubharmonic functions in $\bar{D}$ (plurisubharmonic in a neighborhood of D which may depend on the particular functions). The family of functions plurisubharmonic in a given domain D does not form an algebra. Therefore, there is a natural question of the existence of the Šilov boundary with respect to such a family. However, it has been shown in [20] that the Šilov boundary exists for separating function families which are closed only with respect to addition (or multiplication). Therefore, the existence of the Šilov boundary with respect to plurisubharmonic functions is guaranteed.

Let E be a subset of $\bar{D}$ such that $F \subset E$ and let $b(z)$ be a real bounded function (continuous or not) defined on E. Denoted by $A = A(D,E,b)$ the family of all functions $U(z)$ plurisubharmonic in $\bar{D}$ such that

(1) $U(z) \leqq b(z)$ for $z \in E$,

we define the upper envelope $V^*(z)$ of functions U by

(2) $V(z) = \sup_{U \in A} U(z)$, $V^*(z) = \limsup_{z' \to z} V(z')$, $z, z' \in \bar{D}$.

The function $V^*(z)$ has been introduced and investigated in [4] for the case that D is a pseudoconvex domain with "smooth" boundary and $b(z)$ is defined and continuous on $E = F$. In §8 of [4] the connection of $V^*(z)$ with the envelope of holomorphy of Hartogs domains has been considered. We want to add what follows to these considerations. Let

(3) $$H = \{(z,w) \mid z \in E, \; |w| \leqq e^{-b(z)}\}$$

and

(4) $$G = \{(z,w) \mid z \in D, \; |w| < e^{-V^*(z)}\}.$$

We shall prove

LEMMA 1. *If the functions $a_\nu(z)$, $\nu = 0,1,2,\ldots$, are holomorphic in a neighborhood of $\bar{D}$ and the series*

(5) $$f(z,w) = \sum_{\nu=0}^{\infty} a_\nu(z) w^\nu$$

is uniformly convergent[3] on H, then it is uniformly absolutely convergent in any closed subset of the domain G.

Proof. There is a constant $M > 0$ such that

$$|a_\nu(z) w^\nu| \leqq M \qquad \text{for} \quad (z,w) \in H, \qquad \nu = 0,1,\ldots,$$

i.e.,

$$|a_\nu(z)| \leqq M e^{\nu b(z)} \quad \text{for} \quad z \in E, \qquad \nu = 0,1,\ldots.$$

Since $(1/\nu) \log |a_\nu(z)|$ is plurisubharmonic in $\bar{D}$, therefore by definition of $V^*(z)$ we have

$$\left| \frac{a_\nu(z)}{M} \right|^{1/\nu} \leqq e^{V^*(z)}, \qquad z \in \bar{D}, \qquad \nu = 1,2,\ldots,$$

whence it follows that the series (5) is uniformly absolutely convergent in any compact subset of G.

LEMMA 2. *There exists a sequence $\{V_k(z)\}$ of plurisubharmonic functions in $\bar{D}$ such that*

(6) $$V_k(z) < V(z) \quad \text{for} \quad z \in \bar{D}, \qquad k = 1,2\ldots,$$

and

(7) $$V^*(z) = \{ \sup_{k=1,2,\ldots} V_k(z) \}^*.$$

Proof. Let $\{z_\nu\}$ be an arbitrary sequence of points everywhere dense in D. Denote by $z_{\nu k}$ the point of D such that

(8) $$V^*(z_{\nu k}) = \max_{\|z - z_\nu\| \leqq 1/k} V^*(z).$$

---

(3) It is sufficient to assume that $|a_\nu(z) w^\nu|$, $\nu = 1,2,\ldots$, are uniformly bounded on H.

For any point z_{vk} one may find a sequence $\{z_{vkl}\}_{l=1,2...}$ of points of D such that

$$\lim_{l \to \infty} z_{vkl} = z_{vk} \quad \text{and} \quad \lim_{l \to \infty} V(z_{vkl}) = V^*(z_{vk}).$$

One can easily check that the set D^* of all the points z_{vkl}, $v,k,l = 1,2,...,$ satisfies the following property

(9) $\lim_{z' \to z, z' \in D^*} \sup V(z') = V^*(z), \qquad z \in D.$

Let us arrange the points of D^* into a sequence, say $\{p'_v\}$ and take the sequence $\{p_v\}$ of points

(10) $p'_1, \; p'_1, \; p'_2, \; p'_1, \; p'_2, \; p'_3, \; \cdots .$

By definition of $V(z)$ for any p_k there exists a function $V_k(z)$ plurisubharmonic in $\bar{D}$ such that

(11) $V_k(z) < V(z), \quad z \in \bar{D} \quad \text{and} \quad V_k(z) > V(p_k) - (1/k), \qquad k = 1,2,....$

Since any point of D^* is repeated infinitely many times in the sequence $\{p_v\}$, then

(12) $V_k(z) < V(z), \quad z \in \bar{D} \quad \text{and} \quad V(z) = \sup_k V_k(z), \qquad z \in D^*.$

Therefore, due to (9), the sequence $\{V_k(z)\}$ has all the required properties.

LEMMA 3. *Suppose that for the domain D there exists a sequence of domains of holomorphy $\{D_v\}$ such that*

$$D_v \supset D_{v+1} \supset \bar{D}, \qquad v = 1,2,...,$$

and for any $\varepsilon > 0$ there is v_0 such that

$$D_v \subset D_\varepsilon = \{z| \min_{\zeta \in \bar{D}} \|z - \zeta\| < \varepsilon\} \quad \text{for } v \geqq v_0.$$

Then the function $V^*(z)$ is an upper envelope of all the functions $(1/k) \log |g(z)|$, where k is an integer and $g(z)$ is a function holomorphic in $\bar{D}$ such that

$$(1/k) \log |g(z)| \leqq b(z), \quad \text{for} \quad z \in E.$$

Proof. Without any loss of generality we may assume that the functions $V_k(z)$ defined in the proof of Lemma 2 are plurisubharmonic in D_k, $k = 1,2,...,$ respectively. It is known [4] that

(13) $H_k = \{(z,w)| z \in D_k, \; |w| < \exp[-V_k(z)]\}, \qquad k = 1,2,...,$

is a domain of holomorphy and there exists the function $f_k(z,w)$ holomorphic in H_k such that

(14) $f_k(z,w) = \sum_{v=0}^{\infty} a_v^{(k)}(z)w^v, \qquad (z,w) \in H_k,$

where the $a_v^{(k)}(z)$ are holomorphic in D_k and the series is uniformly convergent in any compact subset of H_k. Moreover,

$$(15) \qquad V_k(z) = \left\{ \limsup_{v \to \infty} \frac{1}{v} \log |a_v^{(k)}(z)| \right\}^*, \qquad z \in D_k.$$

By the uniform convergency of (14) there is a constant $M_k > 0$ which does not depend on v such that

$$|a_v^{(k)}(z)| \leqq M_k \exp\left[v\left(V_k(z) + \frac{1}{2k} \right) \right], \qquad z \in \bar{D}, \qquad v,k = 1,2,\dots.$$

It follows from (15) that

$$\left\{ \sup_v \frac{1}{v} \log \left| \frac{a_v^{(k)}(z)}{M_k e^{1/2k}} \right| \right\}^* = V_k(z).$$

Denoting $g_{kv}(z) = a_v^{(k)}(z)/M_k e^{1/2k})$ we have

$$(16) \qquad \frac{1}{v} \log |g_{kv}(z)| \leqq V_k(z), \qquad z \in \bar{D}, \qquad k,v = 1,2,\dots,$$

and

$$(17) \qquad \left\{ \sup_k \left\{ \sup_v \frac{1}{v} \log |g_{kv}(z)| \right\}^* \right\}^* = V^*(z).$$

To complete the proof it is enough to show that

$$V^*(z) = \left\{ \sup_{k,v} \frac{1}{v} \log |g_{kv}(z)| \right\}^*.$$

Let $A(z) = \sup_{k,v}(1/v) \log |g_{kv}(z)|$. By (12) and (16)

$$(18) \qquad A^*(z) \leqq V^*(z).$$

The function $A^*(z)$ is upper semicontinuous. Therefore, given $z_0 \in \bar{D}$ and $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$A^*(z) \leqq A^*(z_0) + \varepsilon, \quad \text{for} \quad \|z - z_0\| \leqq \delta, \qquad z \in \bar{D}.$$

Thus

$$\frac{1}{v} \log |g_{kv}(z)| \leqq A^*(z) \leqq A^*(z_0) + \varepsilon, \quad \|z - z_0\| < \delta, \quad z \in \bar{D}, \quad k,v = 1,2,\dots,$$

whence

$$V^*(z) = \left\{ \sup_k \left\{ \sup_v \frac{1}{v} \log |g_{kv}(z)| \right\}^* \right\}^* \leqq A^*(z_0) + \varepsilon, \qquad \|z - z_0\| < \delta, \quad z \in \bar{D}.$$

Since $\varepsilon > 0$ is arbitrarily small, we have $V^*(z_0) \leqq A^*(z_0)$. Therefore, $A^*(z) \equiv V^*(z)$, Q.E.D.

Putting $E = \bar{D}$, it follows from Lemma 3

COROLLARY 1. *If D satisfies the assumptions of Lemma 3, then the Šilov boundary of D with respect to functions which are plurisubharmonic in $\bar{D}$ is equal to the Šilov boundary of D with respect to functions which are holomorphic in $\bar{D}$ (compare [4, p. 262]).*

We shall prove also

COROLLARY 2 (LEMMA OF BREMERMANN) [4, p. 256]). *If $U(z)$ is plurisubhar-monic and continuous in a domain of holomorphy G, then for any $\tilde{G} \subset \subset G$ and for any $\varepsilon > 0$ there exist k functions $f_1, ..., f_k$ holomorphic in G and k positive integers $c_1, ..., c_k$ such that*

$$(19) \quad U(z) - \varepsilon \;\leqq\; \sup\left\{\frac{1}{c_1}\log|f_1(z)|, ..., \frac{1}{c_k}\log|f_k(z)|\right\} \;\leqq\; U(z), \quad z \in \tilde{G}.$$

Indeed, there is a domain D which satisfies the conditions of Lemma 3 such that

$$\tilde{G} \subset D \subset \subset G.$$

Therefore, due to the continuity of $U(z)$ there is a finite system of functions $f_i(z)$, $i = 1, 2, ..., k$, which satisfy (19) for $z \in D$. Thus Corollary 2 is true.

THEOREM 1. *If D satisfies the assumptions of Lemma 3, and moreover every function $f(z)$ holomorphic in $\bar{D}$ can be uniformly approximated in $\bar{D}$ by polynomials, then*

$$(20) \qquad\qquad V^*(z) = u^*(z,E,b), \qquad z \in \bar{D}.$$

Proof. By our assumptions $\bar{D}$ is polynomially convex. Therefore, $\Phi(z,E,0) \equiv \Phi(z,\bar{D},0) = 1$ for $z \in \bar{D}$. Thus

$$u(z,E,b) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \log \Phi(z,E,\lambda b), \qquad z \in \bar{D}.$$

Let $g(z)$ be an arbitrary function holomorphic in $\bar{D}$ such that for some positive integer v we have

$$(21) \qquad\qquad |g(z)| \leqq e^{vb(z)}, \qquad z \in E.$$

There is a sequence of polynomials $\{P_k(z)\}$ uniformly convergent to $g(z)$ in $\bar{D}$. We may assume that

$$(22) \qquad\qquad |P_k(z)| \leqq |g(z)|, \qquad z \in \bar{D}, \quad k = 1,2,....$$

Let the degree of P_k be equal to v_k. We have

$$|P_k(z)| \leqq \exp\left[v_k \cdot \frac{v}{v_k} b(z)\right], \qquad z \in E.$$

Therefore, due to Lemma 6.1

$$|P_k(z)| \leqq \Phi^{v_k}\left(z, E, \frac{v}{v_k} b\right) = \left[\Phi^{v_k/v}\left(z, E, \frac{v}{v_k} b\right)\right]^v,$$

whence by (12.1)

$$\frac{1}{v}\log|P_k(z)| \leqq u(z,E,b), \qquad z \in \bar{D}, \quad k = 1,2,... .$$

Then

$$\frac{1}{v}\log|g(z)| \leqq u(z,E,b), \qquad z \in \bar{D}.$$

Since $g(z)$ is an arbitrary holomorphic function satisfying (21), we have by Lemma 3

$$V^*(z) \leqq u^*(z) = u^*(z,E,b), \qquad z \in \bar{D}.$$

But the function $u^*(z)$, being the upper envelope of functions $(1/v) \log |P(z)|$ where $P(z)$ is a polynomial such that $(1/v) \log |P(z)| \leqq b(z)$ for $z \in E$, cannot be larger than $V^*(z)$ at any point of D. The proof is completed.

It will follow from the following theorem that the domain G considered in Lemma 1 cannot be replaced by any larger domain.

THEOREM 2. *If the domain D satisfies the assumptions of Lemma 3, then there exist functions $a_v(z)$, $v = 0, 1, ...$, holomorphic in $\bar{D}$ such that the series*

$$f(z,w) = \sum_{v=0}^{\infty} a_v(z)w^v$$

converges uniformly on H and on any compact subset of G, but it diverges at any point outside of $\bar{G}$.

Proof. It follows from the proof of Lemma 3 that there exists a double sequence of functions $g_{kv}(z)$ holomorphic in $\bar{D}$ such that

(23) $|g_{kv}(z)| \leqq \exp[v\, V^*(z)], \qquad z \in \bar{D}, \qquad k,v = 1,2,...,$

and

(24) $\exp V^*(z) = \{\sup_{k} \{\limsup_{v \to \infty} |g_{kv}(z)|^{1/v}\}\}^*.$

Moreover if

(25) $V_k(z) = \limsup_{v \to \infty} \frac{1}{v} \log |g_{kv}(z)|^{1/v}, \qquad z \in \bar{D}$

and

(26) $U(z) = \sup_{k} V_k(z), \qquad z \in \bar{D},$

then there exists a countable set $D^* \subset D$, everywhere dense in D, such that

(27) $\lim_{z' \to z_0,\ z' \in D^*} \sup U(z') = V^*(z_0).$

Arrange the points of D^* into a sequence

(28) $p'_1,\ p'_1,\ p'_2,\ p'_1,\ p'_2,\ p'_3,\ ...$

and let p_l, $l = 1,2,...$, be the lth point of this sequence. For any l there is n_l such that

$$\limsup_{v \to \infty} \frac{1}{v} \log |g_{n_l v}(p_l)| > U(p_l) - \frac{1}{l}\,, \quad l = 1,2,....$$

Therefore, there is a sequence of positive integers $v_1 < v_2 < ...$ such that

$$\frac{1}{v_l} \log |g_{n_l v_l}(p_l)| > U(p_l) - \frac{2}{l}, \quad l = 1,2,....$$

It follows from (25), (26) and (28) that

$$\limsup_{l \to \infty} \frac{1}{v_l} \log |g_{n_l v_l}(z)| = U(z), \qquad z \in D^*.$$

Therefore, by (27)

(29)
$$\limsup_{l \to \infty} \frac{1}{v_l} \log |g_{n_l v_l}(z)| = V^*(z), \qquad z \in \bar{D}.$$

By assumption there is a seqence of domains of holomorphy $\{D_v\}$ such that

(30)
$$D_v \supset D_{v+1} \supset \bar{D} \quad \text{and} \quad D_v \downarrow \bar{D}.$$

For any $v = 1,2,\ldots$ there is the function $\tilde{g}_v(z)$ whose domain of existence is D_v. Let $\max_{(z,w) \in H} |\tilde{g}_v(z)w^v| = M_v$. The function $g_v(z) = 1/M_v \tilde{g}_v(z)$ is holomorphic in D_v and

$$|g_v(z)w^v| \leq 1, \qquad z \in H,$$

whence

(31)
$$|g_v(z)| \leq \exp[v V^*(z)], \qquad z \in \bar{D}.$$

Denote by $\tilde{a}_v(z)$ the vth function of the sequence

$$\overset{0,\ldots,0}{v_1} g_{n_1 v_1}(z)\,, \quad g_{v_1+1}(z)\,, \quad \overset{0,\ldots,0}{v_2 - v_1 - 2}\,, \quad g_{n_2 v_2}(z)\,, \quad g_{v_2+1}, \, 0, \ldots\,.$$

We claim that the series

(32)
$$\sum_{v=1}^{\infty} a_v(z)w^v\,, \quad \text{where} \quad a_v(z) = \frac{1}{v^2} \tilde{a}_v(z)\,, \qquad v = 1,2,\ldots,$$

has all the required properties. First of all it follows directly from the construction of $a_v(z)$ that the series is uniformly convergent on H. Further by (29) and (31) we have

$$\left\{ \limsup_{v \to \infty} \frac{1}{v} \log |a_v(z)| \right\}^* = V^*(z).$$

Therefore the series (32) is uniformly convergent on any compact subset of G and it is not convergent in a neighborhood of any point (z_0, w_0) such that $z_0 \in D$ and $(z_0, w_0) \notin \bar{G}$. To end the proof it is enough to show that if $(z_0, w_0) \in CG$ and $z_0 \in C\bar{D}$, then the series (32) is divergent. Indeed, by (30) there is some function $a_v(z)$ which is not holomorphic at z_0; therefore, (32) cannot converge at (z_0, w_0).

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